JP.537 Find the angles of a triangle ABC if

$$\frac{\sin A+2\sin B}{\sqrt{\sin^2 B+\sin^2 C+2\cos A\sin B\sin C}}+1=\frac{3\sqrt{3}}{2\sin C}$$

Proposed by Cristian Miu – Romania

Solution 1 by proposer

$h_a=2R\sin B\sin C\,,h_b=2R\sin A\sin C$ so we can write

$$\frac{h_b+2h_a}{\sqrt{b^2+c^2+2bc\cos A}}+\sin C=\frac{3\sqrt{3}}{2}\leftrightarrow$$

$$\frac{h_b+2h_a}{2m_a}+\sin C=\frac{3\sqrt{3}}{2}\leftrightarrow$$

$\frac{F}{bm_a}+\frac{2F}{am_a}+\sin C=\frac{3\sqrt{3}}{2}$ where F is the area of ABC

Let $M\in BC$ such as $BM=MC$ then

$$\frac{F}{bm_a}=\sin\left(\widehat{MAC}\right),\frac{2F}{am_a}=\sin\left(\widehat{AMC}\right)$$

Now we can write

$\sin\left(\widehat{MAC}\right)+\sin\left(\widehat{AMC}\right)+\sin C=\frac{3\sqrt{3}}{2}$. From here we obtain that $C=\frac{\pi}{3}$ and $AM=\frac{BC}{2}$.

But if

$AM=MC$ then $A=\frac{\pi}{2}$.

So, $A=\frac{\pi}{2},B=\frac{\pi}{c},C=\frac{\pi}{3}$

Solution 2 by Marin Chirciu – Romania

In ΔABC non-obtuse we have $\cos A\geq 0$, with equality for $A=90^\circ$

$$\sin^2 B+\sin^2 C+2\cos A\sin B\sin C\geq \sin^2 B+\sin^2 C+2\cdot 0\cdot\sin B\sin C=$$

$$=\sin^2 B+\sin^2 C$$

For $A=90^\circ\Rightarrow B+C=90^\circ\Rightarrow \sin^2 B+\sin^2 C=1$

We obtain:

$$LHS=\frac{\sin A+2\sin B}{\sqrt{\sin^2 B+\sin^2 C+2\cos A\sin B\sin C}}+1\leq\frac{\sin A+2\sin B}{\sqrt{1+2\cdot 0}}+1=$$

$$= \frac{1 + 2\sin B}{1} + 1 = 2\sin B + 2$$

We have $LHS \leq \sin B + 2$

From $RHS = \frac{3\sqrt{3}}{2\sin C} = LHS \geq 2\sin B + 2 \Rightarrow \frac{3\sqrt{3}}{2\sin C} \geq 2\sin B + 2$**, with equality for**

$\sin B = \frac{1}{2}$ **and** $\sin C = \frac{\sqrt{3}}{2} \Rightarrow B = 30^\circ$ **and** $C = 60^\circ$

We deduce that $A = 90^\circ, B = 30^\circ$ **and** $c = 60^\circ$**.**