JP.538. In ΔABC the following relationship holds:

$$\frac{3}{2R}\le\sum\frac{\cos^2\frac{A}{2}}{h_a}\le\frac{3}{4r}$$

Proposed by Alex Szoros – Romania

Solution 1 by proposer, Solution 2 by Marin Chirciu – Romania, Solution 3 by Tapas Das – India

Solution 1 by proposer

Using the formulas

$$\cos A=\frac{b^2+c^2-a^2}{2bc};\ h_a=\frac{2F}{a};\ R=\frac{abc}{4F}\text{ and }16F^2=2\sum a^2b^2-\sum a^4$$

we have:

$$\sum\frac{\cos A}{h_a}=\sum\frac{a}{2F}\left(\frac{b^2+c^2-a^2}{2bc}\right)=\frac{1}{4Fabc}\sum a^2\left(b^2+c^2-a^2\right)=\frac{2\sum a^2b^2-\sum a^4}{16RF^2}=$$

$$=\frac{16F^2}{16RF^2}=\frac{1}{R}\quad(1)$$

Using the identity

$$\sum\frac{1}{h_a}=\frac{1}{r}\quad(2)$$

we can write that

$$\sum\frac{\cos A}{h_a}+\sum\frac{1}{h_a}=\frac{1}{R}+\frac{1}{r}\Rightarrow\sum\frac{1+\cos A}{h_a}=\frac{1}{R}+\frac{1}{r}\Rightarrow 2\sum\frac{\cos^2\frac{A}{2}}{h_a}=\frac{1}{R}+\frac{1}{r}\quad(3)$$

On the other hand from Euler's inequality $R\ge 2r$ we deduce

$$\frac{3}{R}\le\frac{1}{R}+\frac{1}{r}\le\frac{3}{2r}\Rightarrow\frac{3}{2R}\le\sum\frac{\cos^2\frac{A}{2}}{h_a}\le\frac{3}{4r}$$

Solution 2 by Marin Chirciu – Romania

Lemma.
In ΔABC:

$$\sum\frac{\cos^2\frac{A}{2}}{h_a}=\frac{R+r}{2Rr}$$

Proof.

$$\sum\frac{\cos^2\frac{A}{2}}{h_a}=\sum\frac{\frac{s(s-a)}{bc}}{\frac{2S}{a}}=\frac{s}{2S}\sum\frac{a(s-a)}{bc}=\frac{s}{2pr}\cdot\frac{R+r}{R}=\frac{R+r}{2Rr}$$

Let's get back to the main problem.
Using the Lemma we obtain:
RHS.

$$\sum\frac{\cos^2\frac{A}{2}}{h_a}=\frac{R+r}{2Rr}\overset{Euler}{\le}\frac{3}{4r}$$

Equality holds if and only if the triangle is equilateral.
LHS.

$$\sum\frac{\cos^2\frac{A}{2}}{h_a}=\frac{R+r}{2Rr}\overset{Euler}{\ge}\frac{3}{2R}$$

Equality holds if and only if the triangle is equilateral.

Solution 3 by Tapas Das – India

$$\text{WLOG } a\ge b\ge c$$

$$\cos^2\frac{A}{2}\le\cos^2\frac{B}{2}\le\cos^2\frac{C}{2}$$

$$h_a\le h_b\le h_c\Rightarrow\frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c}$$

$$\therefore\sum\frac{\cos^2\frac{A}{2}}{h_a}\overset{Chebyshev}{\le}\frac{1}{3}\cdot\sum\cos^2\frac{A}{2}\sum\frac{1}{h_a}=\frac{1}{3}\left(2+\frac{r}{2R}\right)\cdot\frac{1}{r}\overset{Euler}{\le}\frac{1}{3}\left(2+\frac{1}{4}\right)\cdot\frac{1}{r}=\frac{3}{4r}$$

$$\sum\frac{\cos^2\frac{A}{2}}{h_a}=\frac{1}{2F}\sum a\cos^2\frac{A}{2}\overset{AM-GM}{\ge}\frac{1}{2F}\cdot 3\left[(abc)\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\right]^{\frac{1}{3}}=$$

$$=\frac{1}{2F}\cdot 3\left[4Rrs\cdot\frac{s^2}{16R^2}\right]^{\frac{1}{3}}=\frac{1}{2F}\cdot 3\left[\frac{r^3s^3}{4R^2\cdot r^2}\right]^{\frac{1}{3}}\overset{Euler}{\ge}\frac{1}{2F}\cdot 3\left[\frac{r^3s^3}{4R^2\left(\frac{R}{2}\right)^2}\right]^{\frac{1}{3}}=$$

$$=\frac{1}{2F}\cdot 3\cdot\frac{rs}{R}=\frac{3}{2R}$$