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JP.539. In $\Delta ABC, O \in (AB), OQ \parallel BC$, where $Q \in (AC)$. $P \in (OC)$ such that $RP \parallel BC$, where $R \in (AC)$ and $T \in (AB)$. If the lengths of the segment RT is the geometric mean of the lengths of the segments OQ and BC, then $OP < \frac{OC}{2}$.

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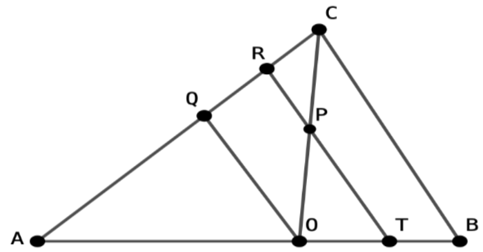
Solution by proposer:



From $OQ \parallel BC \Rightarrow \Delta AOQ \sim \Delta ABC \Rightarrow \frac{AO}{AB} = \frac{QO}{BC} \Rightarrow OQ = \frac{AO \cdot BC}{AB}$

From $RP \parallel BC \Rightarrow \Delta CRP \sim \Delta CQO \Rightarrow \frac{RP}{QO} = \frac{CP}{CO} \Rightarrow RP = \frac{CP \cdot QO}{CO}$

From $PT \parallel BC \Rightarrow \Delta OPT \sim \Delta OCB \Rightarrow \frac{OP}{OC} = \frac{PT}{BC} \Rightarrow PT = \frac{OP \cdot BC}{OC}$

$$RT = RP + PT = \frac{CP \cdot QO + OP \cdot BC}{CO}$$

From $RT^2 = OQ \cdot BC$ (from hypotenuse) $\Rightarrow (CP \cdot QO + OP \cdot BC)^2 = OQ \cdot BC \cdot CO^2$

$$CP \cdot QO + OP \cdot BC = CO \cdot BC \cdot \sqrt{\frac{AO}{AB}},$$

$$CP \cdot \frac{AO \cdot BC}{AB} + OP \cdot BC = CO \cdot BC \cdot \sqrt{\frac{AO}{AB}} |: BC$$

$$CP \cdot AO + OP \cdot AB = CO \cdot \sqrt{AB \cdot AO}, \qquad CP \cdot AO + OP \cdot (AO + OB) = CO\sqrt{AB \cdot AO}$$

$$AO(CP + OP) + OP \cdot OB = CO \cdot \sqrt{AB \cdot AO}$$

$$AO \cdot CO + OP \cdot OB = CO \cdot \sqrt{AB \cdot AO}$$

But $\sqrt{AB \cdot AO} < \frac{AB+AO}{2} \Rightarrow AO \cdot CO + OP \cdot OB < \frac{CO(AB+AO)}{2}$

$$2AO \cdot CO + 2OP \cdot OB < CO \cdot AB + CO \cdot AO$$

$$AO \cdot CO + 2OP \cdot OB < CO \cdot AB, 2OP \cdot OB < CO(AB - AO), 2OP \cdot OB < CO \cdot OB |: OB$$

$$2OP < CO \Rightarrow OP < \frac{OC}{2}$$