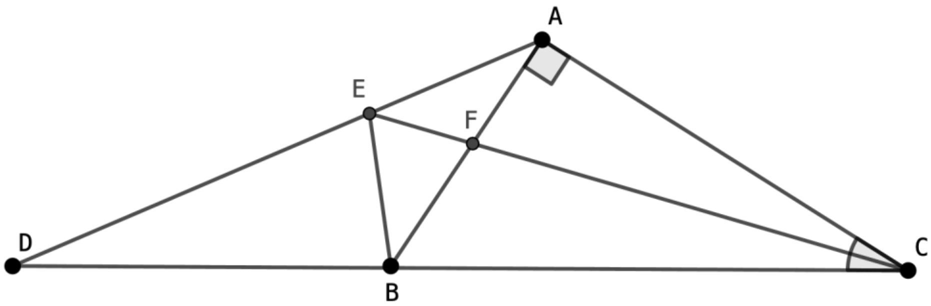
ROMANIAN MATHEMATICAL MAGAZINE

JP.540. Let be ΔACD with $m(\widehat{CAD}) > 90°, B \in (CD)$, such that $m(\widehat{BAC}) = 90°$ and $AC > AB$. The bisector $\widehat{ACD}$ intersects AD in E. If $(BE$ is the bisector $\widehat{ABD}$, prove that:

$$\frac{1}{AD} = \frac{\sqrt{2}}{2}\left(\frac{1}{AB} - \frac{1}{AC}\right)$$

Proposed by Gheorghe Molea – Romania

Solution 1 by proposer



Let be $AB \cap CE = \{F\}$

$(CF =$ the bisector $\widehat{ACB} \Rightarrow \frac{AF}{FB} = \frac{CA}{CB}$

$(BE =$ the bisector $\widehat{ABD} \Rightarrow \frac{DE}{EA} = \frac{BD}{AB}$

$(CE =$ the bisector $\widehat{DCA} \Rightarrow \frac{CD}{CA} = \frac{DE}{EA}$

$$\Rightarrow \frac{CD}{CA} = \frac{BD}{AB} \Leftrightarrow \frac{DB + BC}{CA} = \frac{BD}{AB} \Leftrightarrow \frac{DB}{CA} + \frac{BC}{CA} = \frac{BD}{AB}$$

$$\frac{BD}{AB} - \frac{DB}{AC} = \frac{BC}{CA} \Leftrightarrow BD\left(\frac{1}{AB} - \frac{1}{AC}\right) = \frac{FB}{FA} \Leftrightarrow$$

$$BD\left(\frac{1}{AB} - \frac{1}{AC}\right) = \frac{CB}{CA} \Rightarrow BD = \frac{CB}{CA} \cdot \frac{AB \cdot AC}{AC - AB} \Rightarrow BD = \frac{AB \cdot BC}{AC - AB}$$

$$DC = BD + BC = \frac{AB \cdot BC}{AC - AB} + BC = \frac{BC \cdot AC}{AC - AB}$$

Stewart's relationship in ΔADC:

$$AD^2 \cdot BC + AC^2 \cdot DB - AB^2 \cdot DC = DB \cdot BC \cdot DC$$

$$AD^2 \cdot BC + AC^2 \cdot \frac{AB \cdot BC}{AC - AB} - AB^2 \cdot \frac{BC \cdot AC}{AC - AB} = \frac{AB \cdot BC}{AC - AB} \cdot BC \cdot \frac{BC \cdot AC}{AC - AB}$$

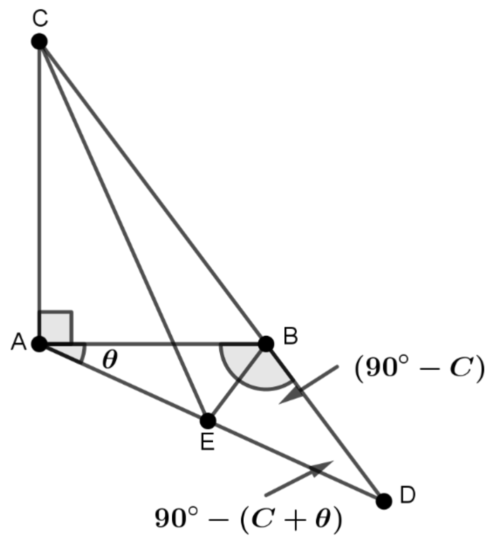
$$AD^2 + AB \cdot AC\left(\frac{AC-AB}{AC-AB}\right) = \frac{AB \cdot AC \cdot BC^2}{(AC-AB)^2}$$

$$AD^2 = AB \cdot AC\left(\frac{BC^2}{(AC-AB)^2} - 1\right) = AB \cdot AC\left(\frac{BC^2 - AC^2 - AB^2 + 2BC \cdot AC}{(AC-AB)^2}\right)$$

$$= AB \cdot AC \cdot \frac{2AC \cdot AB}{(AC-AB)^2} = 2\frac{AB^2 \cdot AC^2}{(AC-AB)^2}$$

$$\Rightarrow AD = \frac{AB \cdot AC\sqrt{2}}{AC-AB} \Rightarrow \frac{1}{AD} = \frac{\sqrt{2}}{2}\left(\frac{AC \cdot AB}{AB \cdot AC}\right) \Rightarrow \frac{1}{AD} = \frac{\sqrt{2}}{2}\left(\frac{1}{AB} - \frac{1}{AC}\right)$$

Solution 2 by Debrata Nag-Kolkata-India



Clearly: $\frac{AE}{ED} = \frac{CA}{CD} = \frac{BA}{BD}$ **(by ∠ - bisector theorem)**

$$\therefore \frac{CA}{CD} = \frac{BA}{BD} = \frac{AC-AB}{BC} \Rightarrow BD = \frac{AB \cdot BC}{AC-AB}$$

Let $\angle BAD = \theta \Rightarrow \angle BDA = 90^\circ - (C+\theta)$ **(1)**

$$\therefore \text{from } \Delta ABD: \frac{AB}{AD} = \frac{\cos(C+\theta)}{\cos C}$$

$$\therefore \frac{AB}{AD} = \cos\theta - \tan C \sin\theta = \left(\cos\theta - \frac{AB}{AC}\sin\theta\right) \quad (2)$$

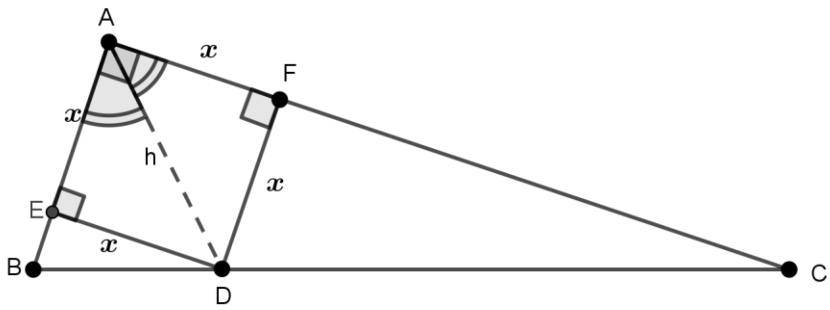
$$\text{Again, } \frac{AB}{BD} = \frac{\cos(C+\theta)}{\sin\theta} = \left(\frac{\cos C}{\tan\theta} - \sin C\right)$$

$$\text{from (1): } \frac{AB}{BD} = \frac{AC-AB}{BC} = \frac{AC}{BC} - \frac{AB}{BC} = \cos C - \sin C$$

$$\Rightarrow \frac{\cos C}{\tan\theta} - \sin C = \cos C = \sin C \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ$$

$$\therefore \text{from (2)}: \frac{AB}{AD} = \frac{1}{\sqrt{2}}\left(\frac{AC-AB}{AC}\right) \Rightarrow \frac{1}{AD} = \frac{\sqrt{2}}{2}\left(\frac{1}{AB} - \frac{1}{AC}\right)$$

Solution 3 by Alin Popa-Romania



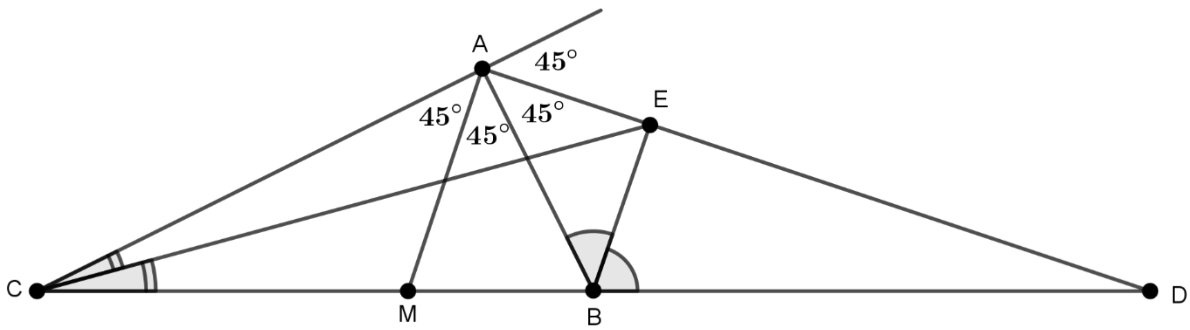
Lemma.

$$A = 90^\circ \Rightarrow h = \frac{h_c}{h+c}\sqrt{2} \Leftrightarrow \frac{1}{h_a} = \left(\frac{1}{h} + \frac{1}{c}\right)\frac{\sqrt{2}}{2}$$

Obs. $AEDF$ **rectangle and as** AD **bisector** $\Rightarrow AEDF$ **square**

Bisector Theorem $\Rightarrow BD = \frac{ac}{h+c}$; $\Delta BDE \sim \Delta BCA \Rightarrow \frac{DE}{AC} = \frac{BD}{BC}$

$$\Leftrightarrow \frac{x}{h} = \frac{ac}{(h+c)a} \Leftrightarrow x = \frac{h_c}{h+c} \Rightarrow h_a = \frac{h_c}{h+c}\sqrt{2}$$



$$\left.\begin{matrix} CE \text{ bisector } \sphericalangle ACD \\ BE \text{ bisector } \sphericalangle ABD \end{matrix}\right\} \Rightarrow AE \text{ exterior bisector } \sphericalangle CAB$$

Let be AM **interior bisector** $\sphericalangle CAB \Rightarrow AB$ **bisector** $\sphericalangle MAD$

$$\left.\begin{array}{l}\dfrac{1}{AB}=\left(\dfrac{1}{AM}+\dfrac{1}{AD}\right)\dfrac{\sqrt{2}}{2}\\ \dfrac{1}{AM}=\left(\dfrac{1}{AB}+\dfrac{1}{AC}\right)\dfrac{\sqrt{2}}{2}\end{array}\right\}\overset{\ominus}{\Longrightarrow}\frac{1}{AD}=\left(\frac{1}{AB}-\frac{1}{AC}\right)\frac{\sqrt{2}}{2}$$