RMM - Geometry Marathon 1401 - 1500

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1401. In any Δ ABC, the fol$l$owing rel$a$tionship hol$d$s :

$$\sum_{cyc}\frac{w_a^k}{b^2+c^2}\ge\frac{(3r)^k}{2R^2}, k\in\mathbb{N}, k\ge 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{w_a^k}{b^2+c^2}=\sum_{cyc}\frac{\left(w_a^{\frac{k}{2}}\right)^2}{b^2+c^2}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}w_a^{\frac{k}{2}}\right)^2}{2\sum_{cyc}a^2}\overset{\text{Leibnitz}}{\ge}\frac{\left(\sum_{cyc}w_a^{\frac{k}{2}}\right)^2}{18R^2}\overset{?}{\ge}\frac{(3r)^k}{2R^2}$$

$$\Leftrightarrow\sum_{cyc}w_a^{\frac{k}{2}}\underset{(*)}{\overset{?}{\ge}}3(3r)^{\frac{k}{2}}$$

$$\text{Now, }\sum_{cyc}w_a^{\frac{k}{2}}\overset{A-G}{\ge}3.\sqrt[3]{\left(\prod_{cyc}w_a\right)^{\frac{k}{2}}}\overset{?}{\ge}3(3r)^{\frac{k}{2}}\Leftrightarrow\left(\prod_{cyc}w_a\right)^{\frac{k}{2}}\overset{?}{\ge}(27r^3)^{\frac{k}{2}}$$

$$\Leftrightarrow\frac{k}{2}.\ln\left(\frac{\prod_{cyc}w_a}{27r^3}\right)\overset{?}{\ge}0\Leftrightarrow\ln\left(\frac{\prod_{cyc}w_a}{27r^3}\right)\overset{?}{\ge}0\ (\because k\in\mathbb{N}, k\ge 2)\Leftrightarrow\prod_{cyc}w_a\underset{(**)}{\overset{?}{\ge}}27r^3$$

$$\text{Again, }\prod_{cyc}w_a\ge\prod_{cyc}h_a=\frac{2r^2s^2}{R}\overset{?}{\ge}27r^3\Leftrightarrow 2s^2\overset{?}{\ge}27Rr\to\text{true}$$

$$\because 2s^2\overset{\text{Gerretsen}}{\ge}27Rr+5r(R-2r)\overset{\text{Euler}}{\ge}27Rr\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\text{ in any }\Delta\text{ ABC},\sum_{cyc}\frac{w_a^k}{b^2+c^2}\ge\frac{(3r)^k}{2R^2}, k\in\mathbb{N}, k\ge 2,$$

$'' ='' $ iff Δ ABC is equil$ateral$ (QED)

1402. AD – altitude in acute $\Delta ABC, DE, DF, O_1, O_2$ – altitudes and circumcenters in $\Delta ABD, \Delta ACD$. Prove that:

$$4\sqrt{[ABD]\cdot[ACD]}\le\left(\sqrt{DE\cdot DO_2}+\sqrt{DF\cdot DO_1}\right)^2\le\frac{RF}{r}$$

Proposed by Radu Diaconu – Romania

Solution by Tapas Das – India

Since ΔABD is right angle triangle

$\therefore O_1$ = circumcentre of ΔABD = Mid point of AB. Similarly, O_2 = Mid point of AC

$$\Delta ABC = \Delta ABD + ADC.$$

$$F = \frac{1}{2}\cdot C\cdot DE + \frac{1}{2}\cdot b\cdot DF \Rightarrow 2F = C\cdot DE + b\cdot DF$$

$$\therefore DO_1 = \frac{c}{2}, \qquad DO_2 = \frac{b}{2}$$

$$4\sqrt{[ABD]\cdot[ACD]} = 4\sqrt{\tfrac{1}{2}AB\cdot DE\cdot\tfrac{1}{2}AC\cdot DF} = 2\sqrt{C\cdot DE\cdot b\cdot DF} \qquad (1)$$

Now $\left(\sqrt{DE\cdot DO_2} + \sqrt{DF\cdot DO_1}\right)^2 \overset{AM-GM}{\geq} 4\left(\sqrt{DE\cdot DO_2\cdot DF\cdot DO_1}\right)$

$$= 4\sqrt{\tfrac{c}{2}\cdot DE\cdot\tfrac{b}{2}\cdot DF} = 2\sqrt{c\cdot DE\cdot b\cdot DF} \qquad (2)$$

From (1) and (2):

$$4\sqrt{[ABD]\cdot[ACD]} \leq \left(\sqrt{DE\cdot DO_2} + \sqrt{DF\cdot DO_1}\right)^2$$

$$\left[\sqrt{DE\cdot DO_2} + \sqrt{DF\cdot DO_1}\right]^2 = \left[\sqrt{DE\cdot\frac{b}{2}} + \sqrt{DF\cdot\frac{c}{2}}\right]^2 = \left[\sqrt{\frac{DE\cdot c}{2}\cdot\frac{b}{c}} + \sqrt{\frac{DF\cdot b}{2}\cdot\frac{c}{b}}\right]^2$$

$$= \left[\sqrt{[ABD]\cdot\frac{b}{c}} + \sqrt{[ABC]\cdot\frac{c}{b}}\right]^2 \overset{Cauchy-Scwarz}{\leq} ([ABD]+[ADC])\left(\frac{b}{c}+\frac{c}{b}\right) =$$

$$= [ABC]\cdot\left(\tfrac{b}{c}+\tfrac{c}{b}\right) \leq F\cdot\tfrac{R}{r}$$

1403. In ΔABC the following relationship holds:

$$\sum\left(\frac{m_a^2}{bc\cdot\csc\frac{B}{2}}\right)^2 \geq \left(\frac{3r}{2R}\right)^3$$

Proposed by Marin Chirciu – Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{m_a^2}{bc\cdot\csc\frac{B}{2}} \geq \sum\frac{s(s-a)}{bc}\cdot\sin\frac{B}{2} =$$

$$= \sum\cos^2\frac{A}{2}\cdot\sin\frac{B}{2} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod\left(\cos^2\frac{A}{2}\cdot\sin\frac{B}{2}\right)} =$$

$$= 3\cdot\sqrt[3]{\underbrace{\prod\cos^2\frac{A}{2}}_{\left(\frac{s}{4R}\right)^2}\cdot\underbrace{\prod\sin\frac{A}{2}}_{\frac{r}{4R}}} = 3\sqrt[3]{\left(\frac{s}{4R}\right)^2\cdot\frac{r}{4R}} =$$

$$= 3\sqrt[3]{\frac{s^2\cdot r}{(4R)^3}} \overset{s^2\ge 27r^2}{\ge} 3\sqrt[3]{\frac{27r^3}{(4R)^3}} = 3\left(\frac{3r}{4R}\right) \qquad (*)$$

$$\sum\left(\frac{m_a^2}{bc\cdot\csc\frac{B}{2}}\right)^2 \overset{(*)}{\ge} 3\left(\frac{3r}{4R}\right)^2 = \frac{3}{4}\left(\frac{3r}{2R}\right)^2 = \frac{3}{4}\left(\frac{3r}{2R}\right)^2 \overset{Euler}{\ge} \frac{3}{4}\cdot\frac{2r}{R}\cdot\left(\frac{3r}{2R}\right)^2 = \left(\frac{3r}{2R}\right)^3$$

Solution 2 by Tapas Das-India

$$\textit{NOTE:-}\ m_a\cdot m_b\cdot m_c \ge \sqrt{r_ar_b}\cdot\sqrt{r_br_c}\cdot\sqrt{r_cr_a} = (r_ar_br_c) = s^2r$$

$$\textit{NOTE:-}\ \prod\sin\frac{A}{2} = \frac{r}{4R}$$

$$\text{Now: }\sum\left(\frac{m_a^2}{bc\cdot\csc\frac{B}{2}}\right)^2 \overset{AM-GM}{\ge} 3\left[\left(\frac{m_am_bm_c}{abc}\right)^2\cdot\prod\sin\frac{A}{2}\right]^{\frac{2}{3}} = 3\left[\left(\frac{s^2r}{4RF}\right)^2\cdot\frac{r}{4R}\right]^{\frac{2}{3}}$$

$$\ge 3\left[\left(\frac{s^2\cdot r}{4R\cdot r\cdot s}\right)^2\cdot\frac{r}{4R}\right]^{\frac{2}{3}} = 3\left[\frac{s^2}{16R^2}\cdot\frac{r}{4R}\right]^{\frac{2}{3}} \ge 3\left[\frac{27r^2\cdot r^2}{64R^3}\right] (\because s^2\ge 27r^2)$$

$$= 3\cdot\frac{9r^2}{16R^2} = \frac{27r^2}{16R^2} = \frac{27r^2\cdot r}{16R^2\cdot r} \overset{Euler}{\ge} \frac{27r^3}{16R^2\cdot\frac{R}{2}} = \left(\frac{3r}{2R}\right)^3$$

1404. In any triangle ABC we have the inequality:

$$\left(\tan\frac{\hat{A}}{2}\tan\frac{\hat{B}}{2}\right)^{\tan\frac{\hat{A}}{2}\tan\frac{\hat{B}}{2}}\cdot\left(\tan\frac{\hat{B}}{2}\tan\frac{\hat{C}}{2}\right)^{\tan\frac{\hat{B}}{2}\tan\frac{\hat{C}}{2}}\cdot\left(\tan\frac{\hat{C}}{2}\tan\frac{\hat{A}}{2}\right)^{\tan\frac{\hat{C}}{2}\tan\frac{\hat{A}}{2}} \le \frac{s^2-2r^2-8Rr}{s^2}$$

Proposed by Radu Diaconu – Romania

Solution by Tapas Das – India

$$\text{Note: }\sum\tan\frac{A}{2} = \frac{4R+r}{s}$$

$$\sum\tan\frac{A}{2}\cdot\tan\frac{B}{2} = 1$$

$$\therefore\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^{\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\cdot\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^{\tan\frac{B}{2}\cdot\tan\frac{C}{2}}\cdot\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^{\tan\frac{C}{2}\tan\frac{A}{2}} \le$$

$$\overset{AM-GM}{\leq}\left[\frac{\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\sum\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\right]^{\sum\tan\frac{A}{2}\cdot\tan\frac{B}{2}}=\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}=1-\frac{2r^2+8Rr}{s^2}$$

$$=\frac{s^2-2r^2-8Rr}{s^2}$$

Note:- $\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}=1-\frac{2r^2+8Rr}{s^2}$

1405. In ΔABC the following relationship holds:

$$9\leq\sum\frac{w_a}{h_a}\sum\frac{h_a}{w_a}\leq 9\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

1st part:

$$\sum\frac{w_a}{h_a}\cdot\sum\frac{h_a}{w_a}\overset{Cauchy-Schwarz}{\geq}(1+1+1)^2=9$$

2nd part: Now

$$\sum\frac{w_a}{h_a}\overset{CBS}{\leq}\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{h_a^2}\right)}\leq\sqrt{\left(\sum r_br_c\right)\cdot\frac{\sum a^2}{4F^2}}\overset{Leibniz}{\leq}\sqrt{\frac{s^2 9R^2}{4r^2s^2}}=\frac{3R}{2r}$$

$$\sum\frac{h_a}{w_a}\leq\sum\frac{h_a}{h_a}\ (\because h_a\leq w_a)\ \text{(analog)}=3$$

$$\therefore\sum\frac{w_a}{h_a}\cdot\sum\frac{h_a}{w_a}\leq\frac{3R}{2r}\cdot 3=9\left(\frac{R}{2r}\right)\leq 9\cdot\frac{R\cdot R}{2rR}\overset{Euler}{\leq}9\frac{R^2}{2r\cdot 2r}=9\left(\frac{R}{2r}\right)^2$$

1406. In ΔABC the following relationship holds:

$$\cos A+\cos B-\cos C+1\geq\frac{3\sqrt{3}r^2}{(s-c)R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$\cos A+\cos B-\cos C+1=4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$=4\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{4\cdot s\cdot(s-a)(s-b)}{abc}=\frac{4s(s-a)(s-b)(s-c)}{abc(s-c)}=\frac{4\cdot s\cdot sr^2}{abc(s-c)}$$

$$=\frac{4\cdot s^2r^2}{4R\cdot r\cdot s\cdot(s-c)}=\frac{sr}{R(s-c)}\overset{s^2\ge 27r^2}{\ge}\frac{3\sqrt{3}r\cdot r}{R(s-c)}=\frac{3\sqrt{3}r^2}{R(s-c)}$$

Note:

$$\cos A+\cos B-\cos C=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-\cos C$$

$$=2\cos\frac{\pi-C}{2}\cos\frac{A-B}{2}-\cos\left(2\cdot\frac{C}{2}\right)=2\sin\frac{C}{2}\cos\frac{A-B}{2}+2\sin^2\frac{C}{2}-1$$

$$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\sin\frac{\pi-(A+B)}{2}\right)-1=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)-1$$

$$=2\sin\frac{C}{2}\cdot 2\cos\frac{A}{2}\cdot\cos\frac{B}{2}-1=4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2}-1$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sin^2\frac{C}{2}=\frac{rc-r}{4R}\qquad (1)$$

$$s\ge 3\sqrt{3}r\qquad (2)$$

$$\cos A+\cos B-\cos C+1\ge\frac{3\sqrt{3}r^2}{(s-c)R}$$

$$\nabla:\sum\cos A+1-2\cos C=\frac{r}{R}+2\cdot 2\sin^2\frac{C}{2}=\frac{r}{R}+4\cdot\sin^2\frac{C}{2}\overset{(1)}{=}$$

$$=\frac{r}{R}+4\cdot\frac{(r_c-r)}{4R}=\frac{r_c}{R}=\frac{F}{(s-c)R}=\frac{s\cdot r}{(s-c)R}\overset{(2)}{\ge}\frac{3\sqrt{3}r^2}{(s-c)R}$$

Solution 3 by Ertan Yildirim-Izmir-Turkiye

Lemma 1: $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$

Lemma 2: $\tan\frac{A}{2}=\frac{r}{s-a}$

$$\cos A+\cos B-\cos C+1=2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)+1-\cos C$$

$$= 2\cdot\sin\frac{C}{2}\cdot\cos\left(\frac{A-B}{2}\right)+2\sin^2\frac{C}{2}=2\cdot\sin\frac{C}{2}\cdot\left(\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right)$$

$$= 2\cdot\sin\frac{C}{2}\cdot\left(\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right)=2\cdot\sin\frac{C}{2}\cdot 2\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}$$

$$= 4\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\cdot\tan\frac{C}{2}=4\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ac}}\cdot\sqrt{\frac{s(s-c)}{ab}}\cdot\frac{r}{s-c}$$

$$= 4\cdot\frac{s\cdot sr}{abc}\cdot r=4\cdot\frac{s\cdot sr}{4R\cdot sr}\cdot\frac{r}{s-c}=\frac{s\cdot r}{(s-c)R}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}r^2}{(s-c)R}$$

Solution 4 by Aissa Hiyab-Morocco

$$\sum\cos A=1+\frac{r}{R}\ \text{(Lemma 1)},\ \sin^2\frac{C}{2}=\frac{c}{s-c}+\frac{r}{4R}\ \text{(Lemma 2)}$$

$$s\geq 3\sqrt{3}r\qquad\text{(Lemma 3)}$$

$$\cos A+\cos B-\cos C+1=\left(\sum\cos A\right)-2\cos C+1$$

$$=\left(1+\frac{r}{R}\right)-2\cos C+1\qquad\text{(Lemma 1)}$$

$$=\frac{r}{R}+2(1-\cos C)=\frac{r}{R}+4\sin^2\frac{C}{2}=\frac{r}{R}+4\times\frac{c}{s-c}\times\frac{r}{4R}\qquad\text{(Lemma 2)}$$

$$=\frac{r}{R}\left(1+\frac{c}{s-c}\right)$$

$$=\frac{r}{R}\times\frac{s}{s-c}\overset{Lemma\ 3}{\leq}\frac{r\times 3\sqrt{3}r}{R\times(s-c)}=\frac{3\sqrt{3}r^2}{(s-c)R}$$

Solution 5 by Soumitra Mandal-India

$$\cos A+\cos B-\cos C+1=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin^2\left(\frac{C}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin^2\left(\frac{\pi-A-B}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\cos^2\left(\frac{A+B}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]$$

$$=2\cos\left(\frac{A+B}{2}\right)\cdot 2\cos\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right)\cos\left(\frac{\frac{A+B}{2}-\frac{A-B}{2}}{2}\right)$$

$$= 4\cos\left(\frac{\pi - C}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2} = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 4\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}} = \frac{4}{c}\cdot\frac{s(s-a)(s-b)}{ab}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{abc}\cdot\frac{1}{s-c} = \frac{4\Delta^2}{abc}\cdot\frac{1}{s-c} = \frac{4s^2r^2}{4Rrs}\cdot\frac{1}{s-c}[\because \Delta = sr]$$

$$= \frac{sr}{R(s-c)} \geq \frac{3\sqrt{3}r^2}{R(s-c)}\left[\because s \geq 3\sqrt{3}r\right] \text{ (proved)}$$

1407. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{\left(r_b^4 + 2r_br_c\left(r_b^2 + r_c^2\right) + r_c^4\right)^3}{r_b^6 + 2r_br_c\left(r_b^4 + r_c^4\right) + r_c^6} \geq \frac{9.6^7r^{11}}{81R^5 - 2560r^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Tapas Das-India

$$\sum\left(r_a^4 + 2r_br_a\left(r_a^2 + r_b^2\right) + r_b^4\right) = \left[2\left(\sum r_a^3\right)\left(\sum r_a\right)\right]$$

$$\sum\left[r_a^6 + 2r_br_a\left(r_a^4 + r_b^4\right) + r_b^6\right] = \left[2\left(\sum r_a^4\right)\left(\sum r_a\right)\right]$$

$$\sum\frac{\left(r_a^4 + 2r_ar_b\left(r_a^2 + r_b^2\right) + r_b^4\right)^3}{r_a^6 + 3r_ar_b\left(r_a^4 + r_b^4\right) + r_b^6} \overset{Holder}{\geq} \frac{1}{3}\cdot\frac{\left[2\left(\sum r_a^3\right)\cdot\left(\sum r_a\right)\right]^3}{2\left(\sum r_a^5\right)\left(\sum r_a\right)} = \frac{4}{3}\cdot\frac{\left(\sum r_a^3\right)^3\cdot\left(\sum r_a\right)^2}{\sum r_a^5}$$

$$\overset{AM-GM}{\geq} \frac{4}{3}\cdot\frac{(3r_ar_br_c)^3(4R+r)^2}{\sum r_a^5} \overset{Euler}{\geq} \frac{4}{3}\cdot\frac{27(s^2r)^3(9r)^2}{\sum r_a^5}$$

$$\overset{Mitrinovic}{\geq} \frac{4}{3}\cdot\frac{27(27r^3)^3(9)^2\cdot r^2}{\sum r_a^5} \geq \frac{4}{3}\cdot\frac{27\cdot 3^9\cdot r^9\cdot 9^2\cdot r^2}{\frac{3^6}{32}(81R^5 - 2560r^5)}$$

$$= (9)\cdot\frac{279936\cdot r^{11}}{(81R^5 - 2560r^5)} = \frac{9\cdot 6^7\cdot r^{11}}{81R^5 - 2560r^5}$$

Now let $a = x + y + z, b = xy + yz + zx, c = xyz$

$$\therefore \sum x^5 = a^5 - 5a^3b + 5ab^2 + 5a^2c - 5bc$$

$$\Rightarrow \sum x^5 = \left(\sum x\right)^5 - 5(x+y)(y+z)(z+x)(x^2 + y^2 + z^2 + xy + yz + zx)$$

$$\therefore \Rightarrow \sum r_a^5 = \left(\sum r_a\right)^5 - 5(r_a + r_b)(r_b + r_c)(r_c + r_a)$$

$$\overset{AM-GM}{\geq} = (4R+r)^5 - 5\cdot(8\cdot r_a r_b r_c)\left(2\cdot 3(r_a r_b r_c)^{\frac{2}{3}}\right)$$

$$\overset{Mitrinovic}{=} (4R+r)^5 - 5\cdot(8\cdot s^2 r)\left(2\cdot 3(s^2 r)^{\frac{2}{3}}\right) \overset{Euler}{\leq} \left(\frac{9R}{2}\right)^5 - 5(8\cdot 3^3 r^3)(2\cdot 3^3 r^2)$$

$$= 3^6\frac{(81R^5 - 2560r^5)}{32}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\left(r_b^4 + 2r_b r_c(r_b^2 + r_c^2) + r_c^4\right)^3}{r_b^6 + 2r_b r_c(r_b^4 + r_c^4) + r_c^6} \overset{Holder}{\geq} \frac{8\left(\sum_{cyc} r_a^4 + \sum_{cyc} r_b r_c(r_b^2 + r_c^2)\right)^3}{6\left(\sum_{cyc} r_a^6 + \sum_{cyc} r_b r_c(r_b^4 + r_c^4)\right)}$$

$$= \frac{4\left(\sum_{cyc} r_a^3\right)^3\left(\sum_{cyc} r_a\right)^3}{3\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_a^5\right)} \overset{Reverse\ Chebyshev}{\geq} \frac{4\left(\sum_{cyc} r_a^3\right)^3\left(\sum_{cyc} r_a\right)^3}{9\left(\sum_{cyc} r_a^6\right)} \overset{Holder}{\geq} \frac{4\left(\sum_{cyc} r_a\right)^9\left(\sum_{cyc} r_a\right)^3}{9^4\left(\sum_{cyc} r_a^6\right)}$$

$$= \frac{4(4R+r)^9(4R+r)^3}{9^4\left(\sum_{cyc} r_a^6\right)} \overset{Euler}{\geq} \frac{4.9^9.r^9.(4R+r)^3}{9^4\left(\sum_{cyc} r_a^6\right)} \overset{?}{\geq} \frac{9.6^7 r^{11}}{81R^5 - 2560r^5}$$

$$= \frac{3^9.2^7.r^{11}}{81R^5 - 2560r^5} \Leftrightarrow \frac{3(81R^5 - 2560r^5)(4R+r)^3}{32r^2} \underset{(*)}{\overset{?}{\geq}} \sum_{cyc} r_a^6$$

$$\text{Now, } \sum_{cyc} r_a^6 = \left(\sum_{cyc} r_a^2\right)^3 - 3\prod_{cyc}(r_b^2 + r_c^2) \overset{Cesaro}{\leq} \left((4R+r)^2 - 2s^2\right)^3 - 24r_a^2 r_b^2 r_c^2$$

$$\overset{Gerretsen}{\leq} \left((4R+r)^2 - 2(16Rr - 5r^2)\right)^3 - 6r^2(2s^2)^2 \overset{Gerretsen}{\leq}$$

$$(16R^2 - 24Rr + 11r^2)^3 - 6r^2(27Rr + 5r(R-2r))^2$$

$$\left(\begin{matrix}\because 16R^2 - 24Rr + 11r^2 = 16R(R-2r) \\ +8Rr + 11r^2 \overset{Euler}{\geq} 8Rr + 11r^2 > 0\end{matrix}\right) \overset{Euler}{\leq} (16R^2 - 24Rr + 11r^2)^3$$

$$-6r^2(27Rr)^2 \overset{?}{\leq} \frac{3(81R^5 - 2560r^5)(4R+r)^3}{32r^2}$$

$$\Leftrightarrow 15552t^8 + 11664t^7 - 128156t^6 + 590067t^5 - 1155072t^4 + 761856t^3$$

$$-1022784t^2 + 186624t - 50272 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(\begin{matrix}15552t^6 + 73872t^5 + 105124t^4 + 715075t^3 \\ +1284732t^2 + 3040484t + 6000224\end{matrix}\right) +12025584\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{Euler}{\geq} 2 \Rightarrow (*) \text{ is true } \therefore \text{ in any } \Delta ABC, \sum_{cyc}\frac{\left(r_b^4 + 2r_b r_c(r_b^2 + r_c^2) + r_c^4\right)^3}{r_b^6 + 2r_b r_c(r_b^4 + r_c^4) + r_c^6}$$

$$\geq \frac{9.6^7 r^{11}}{81R^5 - 2560r^5}, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1408. If $x, y, z > 0$ and $n \in \mathbb{N}$, then in ΔABC holds:

$$\frac{a^{4n-1}\cdot x^{2n}}{h_a}+\frac{b^{4n-1}\cdot y^{2n}}{h_b}+\frac{c^{4n-1}\cdot z^{2n}}{h_c}\geq \frac{3}{2F}\left(\frac{16}{9}F^2\sum yz\right)^n$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\frac{a^{4n-1}\cdot x^{2n}}{h_a}+\frac{b^{4n-1}\cdot y^{2n}}{h_b}+\frac{c^{4n-1}\cdot z^{2n}}{h_c}=\frac{a^{4n}x^{2n}}{2F}+\frac{b^{4n}y^{2n}}{2F}+\frac{c^{4n}z^{2n}}{2F}$$

$$\left(\because h_a=\frac{2F}{a}\right)$$

$$=\frac{1}{2F}[(a^2x)^{2n}+(b^2y)^{2n}+(c^2z)^{2n}]\overset{CBS}{\geq}\frac{1}{2F}\cdot\frac{1}{3^{2n-1}}(a^2x+b^2y+c^2z)^{2n}$$

$$\geq\frac{1}{2F}\cdot\frac{1}{3^{2n-1}}\left(4F\sqrt{xy+yz+zx}\right)^{2n}\text{(Oppenheim)}$$

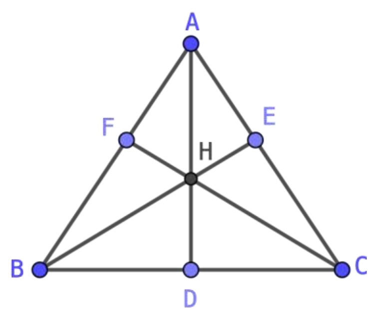
$$=\frac{1}{2F}\cdot\frac{3}{3^{2n}}\left(16F^2\left(\sum xy\right)\right)^n=\frac{3}{2F}\left(\frac{16}{9}\left(\sum xy\right)\right)^n$$

1409. $\Delta DEF, \Delta XYZ$ – are the orthic and the circumcevian triangle of altitudes in acute ΔABC. Prove that:

$$6RF\cdot\sqrt[3]{\frac{2r}{R}\cdot\prod_{cyc}\cos(B-C)}\leq\sum_{cyc}AD\cdot AX\cdot(s-a)\leq s(s^2+r^2-8Rr)$$

Proposed by Radu Diaconu – Romania

Solution by Tapas Das – India



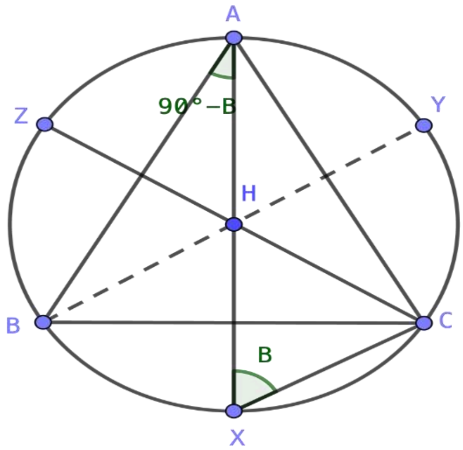
From ΔABD we get

$$\sin B=\frac{AD}{AB}$$

$$\sin B = \frac{AD}{C}$$

$$\therefore AD = c\sin B$$

(analog)



$$\angle BAX = \angle BCX = 90^\circ - B$$

(angles are on the same arc)

$$\therefore \angle ACX = C + 90^\circ - B = 90^\circ - (B - C)$$

From ΔACX we get:

$$\frac{AX}{\sin\angle ACX} = \frac{AC}{\sin\angle AXC}$$

$$[\angle AXC = \angle ABC = B] \Rightarrow \frac{AX}{\sin[90^\circ - (B - C)]} = \frac{b}{\sin B}$$

$$\therefore AX = \frac{b\cos(B-C)}{\sin B} \text{ (analog)}$$

$$\therefore AD \cdot AX \cdot (s-a) = c\sin B \cdot \frac{b\cos(B-C)}{\sin B} \cdot (s-a) = bc(s-a)\cos(B-C)$$

RHS

$$\therefore \sum AD \cdot AX \cdot (s-a) = \sum bc(s-a)\cos(B-C) \le bc(s-a)\cdot 1$$

[Note: $\cos(B-C) \le 1$]

$$= s\left(\sum ab\right) - 3abc = s(s^2 + r^2 + 4Rr) - 12Rrs = s[s^2 + r^2 - 8Rr]$$

LHS

$$\sum AD \cdot AX \cdot (s-a) = \sum bc\,(s-a)\cos(B-C)$$

$$= abc\sum \frac{(s-a)}{a} \cdot \cos(B-C)$$

$$\overset{AM-GM}{\ge} 3abc\left[\frac{(s-a)(s-b)(s-c)}{abc} \cdot \prod \cos(B-C)\right]^{\frac{1}{3}}$$

$$\ge 12RF\left[\frac{sr^2}{4Rrs}\prod \cos(B-C)\right]^{\frac{1}{3}} = 12RF\left[\frac{r}{4R}\prod \cos(B-C)\right]^{\frac{1}{3}}$$

$$= 12RF\left[\frac{2r}{8R}\prod\cos(B-C)\right]^{\frac{1}{3}} = \frac{12RF}{2}\left[\frac{2r}{R}\prod\cos(B-C)\right]^{\frac{1}{3}}$$

$$= 6RF\sqrt[3]{\frac{2r}{R}\prod\cos(B-C)}$$

1410. In ΔABC the following relationship holds:

$$\sum\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{2} + \frac{1}{3\sqrt{3}}\cdot\frac{R}{2r}\sum\sin\frac{A}{2}\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\text{Let } f(x) = \sin\frac{x}{2}, x \in (0,\pi)$$

$$\therefore f'(x) = \frac{1}{2}\cos\frac{x}{2}, \qquad f''(x) = -\frac{1}{4}\sin\frac{x}{2} < 0$$

$\therefore$ f is concave, using Jensen's

$$\therefore \frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \le \sin\left(\frac{A+B+C}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\therefore \sum\sin\frac{A}{2} \le \frac{3}{2}$$

$$\therefore \sum\sin\frac{A}{2}\sin\frac{B}{2} \le \frac{\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2}{3} \le \frac{\left(\frac{3}{2}\right)^2}{3} = \frac{3}{4} \qquad (1)$$

$$\left[\text{Note: } \sum xy \le \frac{(\sum x)^2}{3}\right]$$

Now,

$$\sum\sin\frac{A}{2}\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}} \overset{AM-GM}{\ge} 3\left[\prod\sin\frac{A}{2}\cdot\prod\cos\frac{A}{2}\right]^{\frac{1}{3}} = 3\left[\frac{r}{4R}\cdot\frac{s}{4R}\right]^{\frac{1}{3}}$$

$$\ge 3\left[\frac{r}{4R}\cdot\frac{3\sqrt{3}r}{4R}\cdot\frac{4R}{4R}\right]^{\frac{1}{3}} (\because s \ge 3\sqrt{3}r) \overset{Euler}{\ge} 3\left[\frac{r}{4R}\cdot\frac{3\sqrt{3}r}{4R}\cdot\frac{8r}{4R}\right]^{\frac{1}{3}} = \frac{3\sqrt{3}r\cdot 2}{4R}$$

$$\therefore \frac{1}{2} + \frac{1}{3\sqrt{3}}\cdot\frac{R}{2r}\sum\sin\frac{A}{2}\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}} \ge \frac{1}{2} + \frac{1}{3\sqrt{3}}\cdot\frac{R}{2r}\cdot\frac{3\sqrt{3}r\cdot 2}{4R} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \qquad (2)$$

From (1) and (2) we get the desired result.

1411. Prove that in any triangle ABC holds the inequality

$$\left(\frac{a}{2m_a}\right)^2+\left(\frac{b}{2m_b}\right)^2+\left(\frac{c}{2m_c}\right)^2\ge 1$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Tapas Das-India

$$\text{Let } a>b>c$$

$$\therefore m_a<m_b<m_c$$

$$\therefore \left(\frac{a}{2m_a}\right)^2+\left(\frac{b}{2m_b}\right)^2+\left(\frac{c}{2m_c}\right)^2\ge$$

$$\overset{Chebyshev}{\ge}\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)$$

$$\ge\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\cdot\frac{(1+1+1)^2}{m_a^2+m_b^2+m_c^2}=\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\cdot\frac{9}{\frac{3}{4}(a^2+b^2+c^2)}$$

$$=\frac{1}{3}(a^2+b^2+c^2)\cdot\frac{1}{4}\cdot\frac{3\times 4}{(a^2+b^2+c^2)}=1$$

1412. In any acute Δ ABC, the following relationship holds :

$$8\prod_{cyc}\frac{m_a s_a}{h_a r_a}\le\prod_{cyc}\frac{r_a+r_b}{r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now,}\frac{m_a^2}{s(s-a)}-1\overset{?}{\le}\frac{b^2+c^2}{2bc}-1\Leftrightarrow\frac{(b-c)^2+4s(s-a)-4s(s-a)}{4s(s-a)}\overset{?}{\le}\frac{(b-c)^2}{2bc}$$

$$\Leftrightarrow(b-c)^2.\left(\frac{1}{2bc}-\frac{1}{4s(s-a)}\right)\overset{?}{\ge}0\Leftrightarrow(b-c)^2.\left(\frac{4s(s-a)-2bc}{8sbc(s-a)}\right)\overset{?}{\ge}0$$

$$\Leftrightarrow(b-c)^2.\frac{b^2+c^2-a^2}{8sbc(s-a)}\overset{?}{\ge}0\Leftrightarrow(b-c)^2.\frac{\cos A}{4s(s-a)}\overset{?}{\ge}0\to\text{true}\because\cos A>0$$

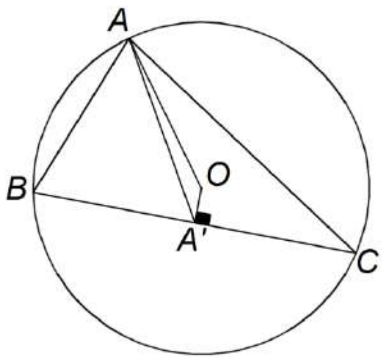
$$\text{in acute triangles} \Rightarrow \frac{m_a^2}{s(s-a)} \le \frac{b^2+c^2}{2bc} \Rightarrow m_a^2 \cdot \frac{2bc}{b^2+c^2} \le s(s-a)$$

$$\Rightarrow \frac{m_a s_a}{h_a r_a} = \frac{m_a^2 \cdot \frac{2bc}{b^2+c^2}}{h_a r_a} \le \frac{s(s-a)}{h_a r_a} = \frac{2R.\, s(s-a)}{bcr_a} \overset{\text{via (i)}}{=} \frac{r_b+r_c}{2r_a} \Rightarrow \frac{2m_a s_a}{h_a r_a} \le \frac{r_b+r_c}{r_a}$$

$$\text{and analogs} \Rightarrow 8\prod_{\text{cyc}} \frac{m_a s_a}{h_a r_a} \le \prod_{\text{cyc}} \frac{r_b+r_c}{r_a} = \prod_{\text{cyc}} \frac{r_a+r_b}{r_c} \therefore \text{ in any acute } \Delta\, ABC,$$

$$8\prod_{\text{cyc}} \frac{m_a s_a}{h_a r_a} \le \prod_{\text{cyc}} \frac{r_a+r_b}{r_c}, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine-Tanger-Morocco



Let A′ be the midpoint of BC and O the circumcenter of ΔABC. In $\Delta AA'O$, we have

$$m_a = AA' \le AO + OA' = R + R\cos A = R(1+\cos A) = 2R\cos^2\frac{A}{2}.$$

Using this result, we have

$$\frac{h_a}{m_a s_a} = \frac{bc}{2R} \cdot \frac{b^2+c^2}{2bcm_a{}^2} = \frac{b^2+c^2}{R.\,4m_a{}^2} = \frac{1}{2R}\left(1+\frac{a^2}{4m_a{}^2}\right) \ge \frac{1}{2R}\left(1+\frac{\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)^2}{4\left(2R\cos^2\frac{A}{2}\right)^2}\right)$$

$$= \frac{1}{2R}\left(1+\tan^2\frac{A}{2}\right) = \frac{1}{2R\cos^2\frac{A}{2}} = \frac{bc}{2Rs(s-a)} = \frac{2r(s-b)(s-c)}{a.\,sr^2} = \frac{2}{\frac{sr}{s-b}+\frac{sr}{s-c}} = \frac{2}{r_b+r_c}.$$

$$\text{Thus,} \quad \frac{2m_a s_a}{h_a r_a} \le \frac{r_b+r_c}{r_a}.$$

Multiplying this inequality with similar ones yields the desired result.

Equality holds if and only if ΔABC is equilateral.

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$8 \cdot \prod \frac{m_a \cdot s_a}{h_a \cdot r_a} \le \prod \frac{r_b+r_c}{r_a}, \; b^2+c^2 \le 4R \cdot m_a \quad (1)$$

$$m_a \leq 2R\cos^2\frac{A}{2}$$

$$4R\cdot\cos^2\frac{A}{2} = \frac{2a}{\sin A}\cdot\cos^2\frac{A}{2} = a\cdot\cot\frac{A}{2} = \frac{a\cdot F}{(s-b)(s-c)} =$$

$$= \frac{(s-b+s-c)\cdot F}{(s-b)(s-c)} = \frac{F}{s-b} + \frac{F}{s-c} = r_b + r_c$$

$$4R\cdot\cos^2\tfrac{A}{2} = r_b + r_c \qquad (3)$$

$$\Rightarrow \frac{2\cdot m_a\cdot s_a}{h_a\cdot r_a} = \frac{2m_a\cdot\dfrac{2bc}{b^2+c^2}\cdot m_a}{h_a\cdot r_a} = \frac{4bc}{h_a}\cdot\frac{m_a^2}{(b^2+c^2)r_a} =$$

$$= \frac{8R}{r_a}\left(\frac{2(b^2+c^2)-a^2}{4(b^2+c^2)}\right) = \frac{2R}{r_a}\cdot\left(2-\frac{a^2}{b^2+c^2}\right) \overset{(1)}{\leq}$$

$$\leq \frac{2R}{r_a}\cdot\left(2-\frac{a^2}{4R\cdot m_a}\right) \overset{(2)}{\leq} \frac{2R}{r_a}\cdot\left(2-\frac{a^2}{8R^2\cdot\cos^2\frac{A}{2}}\right) =$$

$$= \frac{2R}{r_a}\left(2-\frac{\sin^2\frac{A}{2}}{2\cos^2\frac{A}{2}}\right) = \frac{2R}{r_a}\left(2-2\sin^2\frac{A}{2}\right) = \frac{4R\cdot\cos^2\frac{A}{2}}{r_a} \overset{(3)}{=} \frac{r_b+r_c}{r_a}$$

1413. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left(\frac{n_a^2}{w_a^2} + \frac{4r_a}{r_b+r_c}\right) \geq 1 + \frac{4R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a\left(s^2 - s(2s-a) + bc\right) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2$$

$$\Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$

$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= as^2 - 2ah_a r_a \Rightarrow n_a^2 = s^2 - 2h_a r_a \text{ and analogs} \Rightarrow \sum_{cyc}\left(\frac{n_a^2}{w_a^2} + \frac{4r_a}{r_b + r_c}\right)$$

$$= s^2\sum_{cyc}\frac{1}{w_a^2} - \sum_{cyc}\frac{2h_a r_a}{w_a^2} + 4\left(\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{r_b + r_c}\right) - 3\right) \overset{\text{via (i)}}{=}$$

$$s^2\sum_{cyc}\frac{(b+c)^2 a}{4abcs(s-a)} - \sum_{cyc}\frac{4r^2s^2(b+c)^2}{a(s-a).4bcs(s-a)} + 4\left((4R+r)\sum_{cyc}\frac{1}{4R\cos^2\frac{A}{2}} - 3\right)$$

$$= \frac{s^2}{16Rrs^2}\sum_{cyc}\left(\frac{a\left(s^2 + (s-a)^2 + 2s(s-a)\right)}{s-a}\right)$$

$$-\frac{r^2s^2}{4Rrs^2}.\sum_{cyc}\frac{s^2 + (s-a)^2 + 2s(s-a)}{(s-a)^2} + \frac{(4R+r)\left(s^2 + (4R+r)^2\right) - 12Rs^2}{Rs^2}$$

$$= \frac{1}{16Rr}\left(s^2\sum_{cyc}\frac{a - s + s}{s-a} + s(2s) - 2\left(s^2 - 4Rr - r^2\right) + 2s.2s\right)$$

$$-\frac{r}{4R}\left(\frac{1}{r^2}\sum_{cyc} r_a^2 + 3 + \frac{2s\left(4Rr + r^2\right)}{r^2 s}\right) + \frac{(4R+r)\left(s^2 + (4R+r)^2\right) - 12Rs^2}{Rs^2}$$

$$= \frac{1}{16Rr}\left(s^2\left(-3 + \frac{s\left(4Rr + r^2\right)}{r^2 s}\right) + 2\left(4Rr + r^2\right) + 4s^2\right)$$

$$-\frac{r}{4R}\left(\frac{(4R+r)^2 - 2s^2 + 3r^2}{r^2} + \frac{2(4R+r)}{r}\right) + \frac{(4R+r)\left(s^2 + (4R+r)^2\right) - 12Rs^2}{Rs^2}$$

$$= \frac{(2R+5r)s^4 - rs^2\left(32R^2 + 92Rr + 3r^2\right) + 8r^2(4R+r)^3}{8Rr^2s^2} \geq 1 + \frac{4R}{r} = \frac{4R+r}{r}$$

$$\Leftrightarrow (2R+5r)s^4 - rs^2\left(64R^2 + 100Rr + 3r^2\right) + 8r^2(4R+r)^3 \overset{(*)}{\geq} 0 \text{ and}$$

$$\because (2R+5r)\left(s^2 - 16Rr + 5r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \geq (2R+5r)\left(s^2 - 16Rr + 5r^2\right)^2$$

$$\Leftrightarrow (40R - 53r)s^2 \overset{(**)}{\geq} r\left(576R^2 - 846Rr + 117r^2\right) \text{ and again, LHS of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$(40R - 53r)\left(16Rr - 5r^2\right) \overset{?}{\geq} r\left(576R^2 - 846Rr + 117r^2\right)$$

$$\Leftrightarrow 32R^2 - 101Rr + 74r^2 \overset{?}{\geq} 0 \Leftrightarrow (32R - 37r)(R - 2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \text{in any } \Delta\, ABC, \sum_{cyc}\left(\frac{n_a^2}{w_a^2} + \frac{4r_a}{r_b + r_c}\right) \geq 1 + \frac{4R}{r},$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have $w_a^{\ 2} = \frac{4bc}{(b+c)^2}.r_b r_c \leq r_b r_c$.

Also, we have $n_a{}^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s[a^2-(b-c)^2]}{a}$
$= s^2 - \frac{4s(s-b)(s-c)}{a}$
$= s^2 - \frac{4s.sr^2}{a(s-a)} = s^2 - 2h_a r_a$, **then** $n_a{}^2 + 2h_a r_a = s^2$.
And since $\frac{1}{r_b} + \frac{1}{r_c} = \frac{a}{F} = \frac{2}{h_a}$, **then we have** $r_b + r_c = \frac{2r_b r_c}{h_a}$.
Using these results, we have :

$$\frac{n_a{}^2}{w_a{}^2} + \frac{4r_a}{r_b+r_c} \geq \frac{n_a{}^2}{r_b r_c} + \frac{2h_a r_a}{r_b r_c} = \frac{s^2}{r_b r_c} = \frac{r_a}{r} \quad \text{(and analogs)}$$

Therefore,

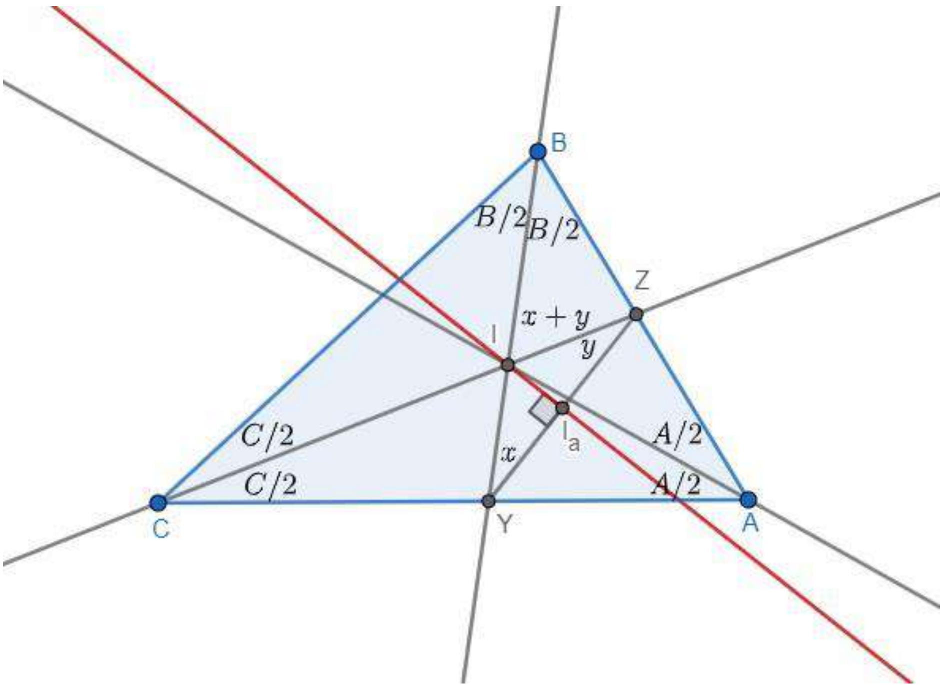
$$\sum_{cyc}\left(\frac{n_a{}^2}{w_a{}^2} + \frac{4r_a}{r_b+r_c}\right) \geq \sum_{cyc}\frac{r_a}{r} = 1 + \frac{4R}{r}.$$

Equality holds if and only if ΔABC **is equilateral.**

1414.

In Δ ABC picturised in the diagram, the following relationship holds

$$2\sqrt{2}\cdot \mathrm{II}_a < IA$$



Proposed by Aissa Hiyab-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via sine law on } \Delta\, AIY, \frac{AI}{\sin\left(C+\frac{B}{2}\right)} = \frac{YI}{\sin\frac{A}{2}}$$

$$\Rightarrow \frac{AI}{\sin\frac{180^\circ - A + C}{2}} = \frac{YI}{\sin\frac{A}{2}} \Rightarrow YI \overset{(i)}{=} \frac{AI\sin\frac{A}{2}}{\cos\frac{A-C}{2}}$$

$$\text{Via sine law on } \Delta\, AIZ, \frac{AI}{\sin\left(B+\frac{C}{2}\right)} = \frac{ZI}{\sin\frac{A}{2}} \Rightarrow \frac{AI}{\sin\frac{180^\circ - A + B}{2}} = \frac{ZI}{\sin\frac{A}{2}} \Rightarrow ZI \overset{(ii)}{=} \frac{AI\sin\frac{A}{2}}{\cos\frac{A-B}{2}}$$

$$\text{Via } \Delta\, II_aY, II_a = YI\sin x \text{ and via } \Delta\, II_aZ, II_a = ZI\sin y \therefore II_a^2 = YI.ZI.\sin x\sin y$$

$$\overset{\text{via (i),(i)}}{\Rightarrow} 8II_a^2 = 8\sin x\sin y.\frac{AI^2\sin^2\frac{A}{2}}{\cos\frac{A-C}{2}.\cos\frac{A-B}{2}} \overset{?}{<} AI^2$$

$$\Leftrightarrow (\cos(x-y)-\cos(x+y)).8\sin^2\frac{A}{2} \overset{?}{<} \cos\frac{2A-(180^\circ - A)}{2}$$

$$+\cos\frac{B-C}{2}\left(\because \frac{-\pi}{2} < \frac{A-C}{2}, \frac{A-B}{2} < \frac{\pi}{2} \Rightarrow \cos\frac{A-C}{2}, \cos\frac{A-B}{2} > 0\right)$$

$$\Leftrightarrow \left(\cos(x-y)-\cos\frac{B+C}{2}\right).8\sin^2\frac{A}{2} \overset{?}{<} \sin\frac{3A}{2} + \frac{b+c}{a}.\sin\frac{A}{2}$$

$$= 3\sin\frac{A}{2} - 4\sin^3\frac{A}{2} + \frac{b+c}{a}.\sin\frac{A}{2}$$

$$\Leftrightarrow \left(\cos(x-y)-\cos\frac{B+C}{2}\right).8\sin\frac{A}{2} \overset{?}{<} 3 - 4\sin^2\frac{A}{2} + \frac{b+c}{a}$$

$$\Leftrightarrow 8\sin\frac{A}{2}.\cos(x-y) - 8\sin^2\frac{A}{2} \overset{?}{<} 3 - 4\sin^2\frac{A}{2} + \frac{b+c}{a}$$

$$\Leftrightarrow \boxed{8\sin\frac{A}{2}.\cos(x-y) \underset{(*)}{\overset{?}{<}} 3 + 4\sin^2\frac{A}{2} + \frac{b+c}{a}}$$

$$\text{Now}, x < x+y = \frac{B+C}{2} = 90^\circ - \frac{A}{2} < 90^\circ \text{ and similarly}, y < 90^\circ \therefore 0 < x < 90^\circ$$

$$\text{and } -90^\circ < -y < 0 \Rightarrow -90^\circ < x-y < 90^\circ \therefore 0 < \cos(x-y) < 1$$

$$\Rightarrow 8\sin\frac{A}{2}.\cos(x-y) < 8\sin\frac{A}{2} \Rightarrow \text{LHS of } (*) \overset{(\bullet)}{<} 8\sin\frac{A}{2} \text{ and } 3 + 4\sin^2\frac{A}{2} + \frac{b+c}{a}$$

$$> 3 + 4\sin^2\frac{A}{2} + 1 \Rightarrow \text{RHS of } (*) \overset{(\bullet\bullet)}{>} 4 + 4\sin^2\frac{A}{2} \therefore (\bullet), (\bullet\bullet) \Rightarrow \text{in order}$$

$$\text{to prove } (*), \text{it suffices to prove} : 4 + 4\sin^2\frac{A}{2} > 8\sin\frac{A}{2} \Leftrightarrow 4\left(1 - \sin\frac{A}{2}\right)^2 > 0$$

$$\rightarrow \text{true} \Rightarrow (*) \text{ is true} \Rightarrow 8II_a^2 < AI^2 \Rightarrow 2\sqrt{2} * II_a < IA\ (QED)$$

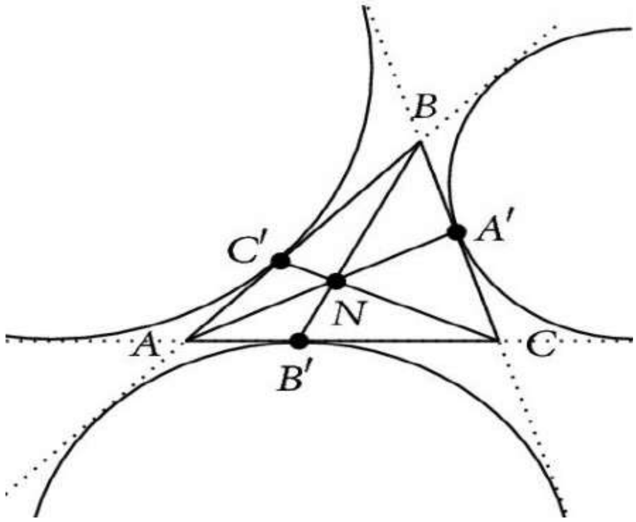
1415. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2 \cdot \sum_{cyc}(s-n_b)(s-n_c) > 4s^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let N be the Nagel's point of ΔABC and AA', BB', CC' be the Nagel's cevians.



Since $AB' = s - c,\ \ AC' = s - b,\ \ BC' = CB' = s - a,$

and using Van Aubel's theorem, we have

$$\frac{NA}{NA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$$

$$\Rightarrow \frac{n_a}{NA} = 1 + \frac{NA}{NA'} = 1 + \frac{s-a}{a} = \frac{s}{a} \Rightarrow NA = \frac{an_a}{s} = 2r.\frac{n_a}{h_a} \text{ (and analogs).}$$

Also, in ΔNBC, we have

$a < NB + NC = 2r\left(\frac{n_b}{h_b} + \frac{n_c}{h_c}\right)$ **(and analogs), (see, for example, Bogdan Fuștei − *About a Few Special Triangles* − www.ssmrmh.ro)**

Now, we have

$$s^2 - {n_a}^2 = s^2 - \left(s(s-a) + \frac{s(b-c)^2}{a}\right) = \frac{s[a^2 - (b-c)^2]}{a} = \frac{4s(s-b)(s-c)}{a}$$

$$= \frac{4s.sr^2}{a(s-a)} = 2r_a h_a.$$

Using these results, we have,

$$s - n_a = \frac{s^2 - {n_a}^2}{s + n_a} = \frac{2r_a h_a}{s + n_a} = \frac{2r_a}{\frac{a}{2r} + \frac{n_a}{h_a}} > \frac{2r_a}{\left(\frac{n_b}{h_b} + \frac{n_c}{h_c}\right) + \frac{n_a}{h_a}}$$

$$\Rightarrow\left(\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\right)(s-n_a)>2r_a\ \text{(and analogs)}.$$

Therefore

$$\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2\cdot\sum_{cyc}(s-n_b)(s-n_c)\geq\sum_{cyc}2r_b.2r_c=4s^2.$$

1416. In any Δ ABC, Δ A′B′C′ the following relationship holds:

$$\min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}$$
$$\geq 8+\max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2w_a}{w_b+w_c}.1.1}\overset{G-H}{\geq}\frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{6w_a}{w_b+w_c}}{\frac{2w_a}{w_b+w_c}+\frac{2w_a}{w_b+w_c}+1}$$

$$=\frac{6}{\sqrt[3]{2}}\sum_{cyc}\frac{w_a^2}{4w_a^2+w_aw_b+w_aw_c}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \sum_{cyc}w_aw_b\leq\sum_{cyc}w_a^2}}{\geq}\frac{1}{\sqrt[3]{2}}\cdot\frac{6\left(\sum_{cyc}w_a\right)^2}{4\sum_{cyc}w_a^2+2\sum_{cyc}w_a^2}$$

$$\overset{w_a\geq h_a=\frac{2rs}{a}\text{ and analogs}}{\geq}\frac{1}{\sqrt[3]{2}}\cdot\frac{\left(2rs.\sum_{cyc}\frac{1}{a}\right)^2}{\sum_{cyc}s(s-a)}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Mitrinovic}}}{\geq}\frac{1}{\sqrt[3]{2}}\cdot\frac{4\left(2rs.\frac{9}{2s}\right)^2}{27R^2}$$

$$\Rightarrow\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\blacksquare)}{\geq}\frac{1}{\sqrt[3]{2}}\cdot\frac{12r^2}{R^2}\overset{?}{\geq}\frac{3}{\sqrt[3]{2}}-\frac{2(R^2-4r^2)}{r^2}$$

$$\Leftrightarrow\frac{2(R^2-4r^2)}{r^2}\overset{?}{\geq}\frac{3}{\sqrt[3]{2}}\left(1-\frac{4r^2}{R^2}\right)=\frac{3}{\sqrt[3]{2}}\cdot\frac{R^2-4r^2}{R^2},\text{ and proving it will be complete}$$

if we can show : $2.\sqrt[3]{2}.R^2>3r^2\ \left(\because R^2-4r^2\overset{\text{Euler}}{\geq}0\right)\to\text{true}$

$$\because 2.\sqrt[3]{2}.R^2\overset{\text{Euler}}{\geq}8.\sqrt[3]{2}.r^2>3r^2\ \therefore\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\bullet)}{\geq}\frac{3}{\sqrt[3]{2}}-\frac{2(R^2-4r^2)}{r^2}$$

$$\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2m_a}{m_b+m_c}.1.1}\overset{G-H}{\geq}\frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{6m_a}{m_b+m_c}}{\frac{2m_a}{m_b+m_c}+\frac{2m_a}{m_b+m_c}+1}$$

$$=\frac{6}{\sqrt[3]{2}}\sum_{cyc}\frac{m_a^2}{4m_a^2+m_am_b+m_am_c}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \sum_{cyc}m_am_b\le\sum_{cyc}m_a^2}}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{6\left(\sum_{cyc}m_a\right)^2}{4\sum_{cyc}m_a^2+2\sum_{cyc}m_a^2}$$

$$\overset{m_a\ge h_a=\frac{2rs}{a}\text{ and analogs}}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{\left(2rs.\sum_{cyc}\frac{1}{a}\right)^2}{\frac{3}{4}\sum_{cyc}a^2}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Leibnitz}}}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{4\left(2rs.\frac{9}{2s}\right)^2}{27R^2}$$

$$\Rightarrow\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\overset{(\blacksquare\blacksquare)}{\ge}\frac{1}{\sqrt[3]{2}}\cdot\frac{12r^2}{R^2}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}-\frac{4(R-2r)}{r}$$

$$\Leftrightarrow\frac{4(R-2r)}{r}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}\cdot\frac{(R-2r)(R+2r)}{R^2}$$ *and proving it will be complete*

if we can show : $4R^2>3r(R+2r)\ \left(\because R^2-4r^2\overset{\text{Euler}}{\ge}0\text{ and }\sqrt[3]{2}>1\right)$

$$\Leftrightarrow(R-2r)(4R+5r)+4r^2>0\to\textbf{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\therefore\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\bullet\bullet)}{\ge}\frac{3}{\sqrt[3]{2}}-\frac{4(R-2r)}{r}$$

$$\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2w_a}{w_b+w_c}.1.1}\overset{A-G}{\le}\frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{2w_a}{w_b+w_c}+2}{3}$$

$$=\frac{2}{3}\cdot\frac{1}{\sqrt[3]{2}}\left(\sum_{cyc}w_a\right)\left(\sum_{cyc}\frac{1}{w_b+w_c}\right)\overset{A-G}{\le}\frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}(\sqrt{3}s)\left(\sum_{cyc}\frac{1}{\sqrt{w_bw_c}}\right)\overset{\substack{w_a\ge h_a\text{ and analogs + CBS}\\ \text{and}\\ \text{Mitrinovic}}}{\le}$$

$$\frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}\cdot\sqrt{3}.\frac{3\sqrt{3}R}{2}\cdot\sqrt{\sum_{cyc}\frac{1}{h_a}}\cdot\sqrt{\sum_{cyc}\frac{1}{h_a}}\Rightarrow\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\blacksquare\blacksquare\blacksquare)}{\le}$$

$$\frac{3}{2.\sqrt[3]{2}}\cdot\frac{R}{r}\overset{?}{\le}\frac{3}{\sqrt[3]{2}}+\frac{4(R-2r)}{r}\Leftrightarrow\frac{4(R-2r)}{r}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}\cdot\frac{R-2r}{2r}$$ *and proving it will be*

complete if we can show : $8.\sqrt[3]{2}>3\ \left(\because R-2r\overset{\text{Euler}}{\ge}0\text{ and }\sqrt[3]{2}>1\right)\to\textbf{true}$

$$\therefore\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}\overset{(\bullet\bullet\bullet)}{\le}\frac{3}{\sqrt[3]{2}}+\frac{4(R-2r)}{r}$$

$$\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2m_a}{m_b+m_c}.1.1}\overset{A-G}{\le}\frac{1}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{2m_a}{m_b+m_c}+2}{3}$$

$$=\frac{2}{3}\cdot\frac{1}{\sqrt[3]{2}}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)\overset{A-G}{\le}\frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}(4R+r)\left(\sum_{cyc}\frac{1}{\sqrt{m_bm_c}}\right)$$

$$\overset{m_a\geq h_a \text{ and analogs + CBS and Euler}}{\leq} \frac{1}{3}\cdot\frac{1}{\sqrt[3]{2}}\left(\frac{9R}{2}\right)\cdot\sqrt{\sum_{cyc}\frac{1}{h_a}}\cdot\sqrt{\sum_{cyc}\frac{1}{h_a}}$$

$$\Rightarrow \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}} \overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{\leq} \frac{3}{2.\sqrt[3]{2}}\cdot\frac{R}{r} \overset{?}{\leq} \frac{3}{\sqrt[3]{2}}+\frac{2(R^2-4r^2)}{r^2}$$

$$\Leftrightarrow \frac{2(R^2-4r^2)}{r^2} \overset{?}{\geq} \frac{3}{\sqrt[3]{2}}\cdot\frac{R-2r}{2r} \text{ and proving it will be complete if we can show :}$$

$$4(R+2r) > 3r\left(\because R-2r \overset{\text{Euler}}{\geq} 0 \text{ and } \sqrt[3]{2}>1\right) \Leftrightarrow 4R+5r>0 \to \text{true}$$

$$\therefore \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}} \overset{(\bullet\bullet\bullet)}{\leq} \frac{3}{\sqrt[3]{2}}+\frac{2(R^2-4r^2)}{r^2}$$

$$\text{Again, } \frac{R^2R'}{r^2r'} = \frac{(R^2-4r^2+4r^2)(R'-2r'+2r')}{r^2r'}$$

$$= \frac{2r'(R^2-4r^2)+4r^2(R'-2r')+8r^2r'+(R^2-4r^2)(R'-2r')}{r^2r'}$$

$$= 8+\frac{2(R^2-4r^2)}{r^2}+\frac{4(R'-2r')}{r'}+\frac{(R^2-4r^2)(R'-2r')}{r^2r'}$$

$$\overset{\text{Euler}}{\geq} 8+\frac{2(R^2-4r^2)}{r^2}+\frac{4(R'-2r')}{r'} \therefore \frac{R^2R'}{r^2r'} \overset{(\text{?})}{\geq} 8+\frac{2(R^2-4r^2)}{r^2}+\frac{4(R'-2r')}{r'}$$

$$\text{Let } m = \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}, \sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\} \text{ and}$$

$$M = \max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}}, \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$$\boxed{Case\ 1}\ m = \sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}; M = \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}} \text{ and then, via } (\blacksquare), (\blacksquare\blacksquare\blacksquare\blacksquare),$$

$$LHS - RHS \geq \frac{1}{\sqrt[3]{2}}\cdot\frac{12r^2}{R^2}-\frac{3}{2.\sqrt[3]{2}}\cdot\frac{R}{r}+\frac{R^2R'}{r^2r'}-8 \overset{\text{Euler}}{\geq} -\frac{3}{\sqrt[3]{2}}\left(\frac{R}{2r}-\frac{4r^2}{R^2}\right)$$

$$+\frac{2(R^2-4r^2)}{r^2} \overset{?}{\geq} 0 \Leftrightarrow \frac{2(R^2-4r^2)}{r^2} \overset{?}{\geq} \frac{3}{\sqrt[3]{2}}\cdot\frac{R^3-8r^3}{2R^2r} \text{ and proving it will be complete}$$

$$\text{if we can show : } 4R^2(R+2r) > 3r(R^2+4r^2-2Rr)$$

$$\left(\because R-2r \overset{\text{Euler}}{\geq} 0 \text{ and } \sqrt[3]{2}>1\right) \Leftrightarrow 4R^3+5R^2r+6r^2(R-2r)>0 \to \text{true}$$

$$\because R \overset{\text{Euler}}{\geq} 2r \therefore \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}, \sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\geq 8+\max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}}, \sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$\boxed{Case\ 2}$ $m=\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}; M=\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}}$ *and then, via* (■■), (■■■),

$$LHS-RHS\ge\frac{1}{\sqrt[3]{2}}\cdot\frac{12r'^2}{R'^2}-\frac{3}{2.\sqrt[3]{2}}\cdot\frac{R'}{r'}+\frac{R^2R'}{r^2r'}-8\overset{Euler}{\ge}-\frac{3}{\sqrt[3]{2}}\left(\frac{R'}{2r'}-\frac{4r'^2}{R'^2}\right)$$

$$+\frac{4(R'-2r')}{r'}\overset{?}{\ge}0\Leftrightarrow\frac{4(R'-2r')}{r'}\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}\cdot\frac{R'^3-8r'^3}{2R'^2r'}$$ *and proving it will be complete if we can show* : $8R'^2>3(R'^2+4r'^2-2R'r')$

$$\left(\because R'-2r'\overset{Euler}{\ge}0\ and\ \sqrt[3]{2}>1\right)\Leftrightarrow 5R'^2+6r'(R'-2r')>0\to true$$

$$\because R'\overset{Euler}{\ge}2r'\therefore \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\ge 8+max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$\boxed{Case\ 3}$ $m=\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}}; M=\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}}$, then, via (•), (•••), (⍰),

$$LHS-RHS\ge\frac{3}{\sqrt[3]{2}}-\frac{2(R^2-4r^2)}{r^2}+8+\frac{2(R^2-4r^2)}{r^2}+\frac{4(R'-2r')}{r'}-8-\frac{3}{\sqrt[3]{2}}$$

$$-\frac{4(R'-2r')}{r'}=0\therefore \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\ge 8+max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

$\boxed{Case\ 4}$ $m=\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}; M=\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}$, then, via (••), (••••), (⍰),

$$LHS-RHS\ge\frac{3}{\sqrt[3]{2}}-\frac{4(R'-2r')}{r'}+8+\frac{2(R^2-4r^2)}{r^2}+\frac{4(R'-2r')}{r'}-8-\frac{3}{\sqrt[3]{2}}$$

$$-\frac{2(R'^2-4r'^2)}{r'^2}=0\therefore \min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\ge 8+max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$ $\therefore$ combining *all* cases,

in *any* $\Delta\ ABC, \Delta\ A'B'C'$,

$$\min\left\{\sum_{cyc}\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum_{cyc}\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}$$

$$\geq 8$$

$$+\max\left\{\sum_{cyc}\sqrt[3]{\frac{w_a'}{w_b'+w_c'}},\sum_{cyc}\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}, ''='' \text{ iff } \Delta\,ABC, \Delta\,A'B'C' \text{ are each equilateral (QED)}$$

1417. In acute ΔABC the following relationship holds:

$$\sqrt{3}\sum_{cyc}\sec A+9\sum_{cyc}\csc A\geq 24\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$f(x)=\frac{1}{\cos x}\Rightarrow f'(x)=\frac{\sin x}{\cos^2 x}>0\ (\forall)x\in\left[0;\frac{\pi}{2}\right]$$

$$f''(x)=\frac{\cos^3 x+2\cos x\sin^2 x}{\cos^4 x}>0\ (\forall)x\in\left[0;\frac{\pi}{2}\right]\Rightarrow$$

$$\Rightarrow f\rightarrow\text{convex on }\left[0;\frac{\pi}{2}\right]\Rightarrow\frac{f(A)+f(B)+f(C)}{3}\overset{Jensen}{\geq}f\left(\frac{A+B+C}{3}\right)\Rightarrow$$

$$\Rightarrow\sum\sec A\geq 3\sec\frac{\pi}{3}=\frac{3}{\cos\frac{\pi}{3}}=\frac{3}{\frac{1}{2}}=6\quad(1)$$

$$g(x)=\csc x=\frac{1}{\sin x}\Rightarrow g'(x)=\frac{-\cos x}{\sin^2 x}\Rightarrow g''(x)=\frac{\sin^3 x+2\sin x\cos^3 x}{\sin^4 x}>0$$

$$\Rightarrow g(x)\rightarrow\text{convex on }\left[0;\frac{\pi}{2}\right]\Rightarrow\frac{g(A)+g(B)+g(C)}{3}\geq g\left(\frac{A+B+C}{3}\right)$$

$$\Rightarrow\sum\csc A\geq 3\frac{1}{\sin\frac{\pi}{3}}=3\cdot\frac{1}{\frac{\sqrt{3}}{2}}=\frac{6}{\sqrt{3}}=\frac{6\sqrt{3}}{3}=2\sqrt{3}\quad(2)$$

From (1) and (2) $\Rightarrow\sqrt{3}\sum\sec A+9\sum\csc A\geq 6\sqrt{3}+18\sqrt{3}=24\sqrt{3}$

Solution 2 by Marin Chirciu-Romania

Lemma: In ΔABC holds:

$$\sum\sec A\geq 6$$

$$\sum\csc A\geq 2\sqrt{3}$$

Proof.

$$\sum\sec A=\sum\frac{1}{\cos A}=\frac{p^2+r^2-4R^2}{p^2-(2R+r)^2}\overset{Gerretsen}{\geq}6,$$

$$\sum \csc A = \sum \frac{1}{\sin A} = \frac{p^2 + r^2 + 4Rr}{2pr} \underset{Mitrinovic}{\overset{Gerretsen}{\geq}} 2\sqrt{3}.$$

Let's get back to the main problem.Using the Lemma we obtain:

$$LHS = \sqrt{3}\sum \sec A + 9\sum \csc A \geq 24\sqrt{3} \overset{Lemma}{\geq} \sqrt{3}\cdot 6 + 9\cdot 2\sqrt{3} = 24\sqrt{3}$$

Equality holds if and only if the triangle is equilateral.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For acute triangle ABC

$$\sqrt{3}(\sec A + \sec B + \sec C) + 9(\csc A + \csc B + \csc C)$$

$$= \sqrt{3}\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right) + 9\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)$$

$$\geq \sqrt{3}(6) + 9\left(2\sqrt{3}\right) = \sqrt{3}(24) \text{ ok}$$

Because $\cos A + \cos B + \cos C \leq \frac{3}{2}$

$$\Rightarrow 1 \leq \frac{3}{2}\cdot\frac{1}{\cos A + \cos B + \cos C} \Rightarrow 1 \leq \frac{3}{2}\left(\frac{1}{9}\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right)\right)$$

$$\Rightarrow 6 \leq \frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}$$

and $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2} \Rightarrow 1 \leq \frac{3\sqrt{3}}{2}\left(\frac{1}{\sin A + \sin B + \sin C}\right)$

$$\Rightarrow 1 \leq \frac{3\sqrt{3}}{2}\left(\frac{1}{9}\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}\right)\right) \Rightarrow 2\sqrt{3} \leq \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C}$$

Therefore it is to be true.

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\sum \sec A \geq 3\sqrt[3]{\prod \sec A} \geq 3\sqrt[3]{\sec^3\frac{\pi}{3}} = 6$$

$$\sum \csc A \geq 3\sqrt[3]{\prod \csc A} \geq 3\sqrt[3]{\csc\frac{\pi}{3}} = 2\sqrt{3} \Rightarrow LHS = \sqrt{3}\cdot 6 + 9\cdot 2\sqrt{3} = 24\sqrt{3}$$

Therefore $\Rightarrow \sqrt{3}\sum_{cyc}\sec A + 9\sum_{cyc}\csc A \geq 24\sqrt{3}$

1418. **In any ΔABC the following relationship holds:**

$$\sqrt{\frac{a^2+bc}{b^2+ac}}+\sqrt{\frac{b^2+ac}{c^2+ab}}+\sqrt{\frac{c^2+ab}{a^2+bc}}+\frac{R^2}{4r^2}\geq 1+\frac{2a}{b+c}+\frac{2b}{a+c}+\frac{2c}{a+b}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Tapas Das-India

First of all, we can easily prove the following equality

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}$$

Consequently, it is sufficient to prove

$\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}\leq\frac{R}{6r}+\frac{F}{6}$. Let $f(s^2)=\frac{R}{6r}+\frac{7}{6}-\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}$

f is a decreasing function and we need to prove $f(s^2)\geq 0$

Applying Gerretsen's inequality we deduce that: $s^2\leq 4R^2+4Rr+3r^2$

Therefore, it is sufficient to prove $f(4R^2+4Rr+3r^2)\geq 0$

After some simplification, we get

$$(R-2r)(2(R-2r)+5(R-2r)+r^2)\geq 0$$

(Euler)

$$\therefore\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{R}{6r}+\frac{7}{6}$$

We need to show: $1+2\left(\frac{R}{6r}+\frac{7}{6}\right)\leq\frac{R^2}{4r^2}+3$

or $\frac{R^2}{4r^2}+3-1-2\left(\frac{R}{6r}+\frac{7}{6}\right)\geq 0$ or $\frac{R^2}{4r^2}-\frac{R}{3r}-\frac{1}{3}\geq 0$

or $(R-2r)(3R+2r)\geq 0$ (True) Euler

Note: $\sqrt{\frac{a^2+bc}{b^2+ac}}+\sqrt{\frac{b^2+ac}{c^2+ab}}+\sqrt{\frac{c^2+ab}{a^2+bc}}\geq 3$

By (AM-GM)

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

1) $\sum\sqrt{\frac{a^2+bc}{b^2+ca}}+\frac{R^2}{4r^2}\overset{AM\geq GM}{\geq}3+\frac{R^2}{4r^2}$ (1)

$$2)\ 1+2\sum\frac{a}{b+c}=1+\frac{1}{2}\sum a\cdot\frac{4}{b+c}\le 1+\frac{1}{2}\sum a\left(\frac{1}{b}+\frac{1}{c}\right)$$

$$=1+\frac{1}{2abc}\cdot\sum a^2(b+c)=1+\frac{\sum a\sum ab-3abc}{2abc}=$$

$$=1-\frac{3}{2}+\frac{\sum a\cdot\sum ab}{2abc}=-\frac{1}{2}+\frac{2p(p^2+4Rr+r^2)}{8p\cdot Rr}\le$$

$$\overset{Gerretsen}{\le}-\frac{1}{2}+\frac{4R^2+8Rr+4r^2}{4Rr}=2-\frac{1}{2}+\frac{R}{r}+\frac{r}{R}=$$

$$=\frac{3}{2}+\frac{R}{r}+\frac{r}{R}\overset{?}{\le}3+\frac{R^2}{4r^2}$$

$$\frac{R}{r}=y$$

$$\frac{y^2}{4}+\frac{3}{2}-y-\frac{1}{y}\ge 0$$

$$y^3-4y^2+6y-4\ge 0$$

$$(y-2)(y^2-2y+2)=\underbrace{(y-2)}_{\ge 0}\underbrace{\left((y-1)^2+1\right)}_{>0}\ge 0$$

1419. In any Δ ABC, the following relationship holds :

$$\min\left\{\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\right\}+\frac{R+\sqrt{3}s}{r}\ge 11+2\cdot\max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\overset{A-G}{\ge}3\ \therefore\ \min\left\{\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\right\}\ge 3$$

$$\therefore\ \text{it suffices to prove :}\ \boxed{\frac{R+\sqrt{3}s}{r}\overset{(\bullet)}{\ge}8+2.\max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\}}$$

$$\text{Now, }8+2\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{A-G}{\le}8+\frac{\sum_{cyc}a^3}{abc}=8+\frac{2s(s^2-6Rr-3r^2)}{4Rrs}$$

$$=\frac{s^2+10Rr-3r^2}{2Rr}\overset{?}{\le}\frac{R+\sqrt{3}s}{r}\Leftrightarrow\frac{R^2+3s^2+2\sqrt{3}Rs}{r^2}\underset{(*)}{\overset{?}{\ge}}\frac{(s^2+10Rr-3r^2)^2}{4R^2r^2}$$

$$\text{Now, LHS of } (*) \overset{\text{Mitrinovic}}{\geq} \frac{R^2+3s^2+2\sqrt{3}s.\frac{2s}{3\sqrt{3}}}{r^2}$$

$$= \frac{3R^2+13s^2}{3r^2} \overset{?}{\geq} \frac{(s^2+10Rr-3r^2)^2}{4R^2r^2} \Leftrightarrow 4R^2(3R^2+13s^2) \overset{?}{\geq} 3(s^2+10Rr-3r^2)^2$$

$$\Leftrightarrow 12R^4-300R^2r^2+180Rr^3-27r^4+(52R^2-60Rr+18r^2)s^2 \underset{(**)}{\overset{?}{\geq}} 3s^4$$

$$\text{Now, } 3s^4 \overset{\text{Gerretsen}}{\leq} (12R^2+12Rr+9r^2)s^2$$

$$\overset{?}{\leq} 12R^4-300R^2r^2+180Rr^3-27r^4+(52R^2-60Rr+18r^2)s^2$$

$$\Leftrightarrow (40R^2-72Rr+9r^2)s^2+12R^4-300R^2r^2+180Rr^3-27r^4 \underset{(***)}{\overset{?}{\geq}} 0$$

$$\text{Again, LHS of } (***) \overset{\text{Gerretsen}}{\geq}$$

$$(40R^2-72Rr+9r^2)(16Rr-5r^2)+12R^4-300R^2r^2+180Rr^3-27r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3t^4+160t^3-413t^2+171t-18 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(3t^3+125t^2+41t(t-2)+t+9) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \boxed{\frac{R+\sqrt{3}s}{r} \overset{(\blacksquare)}{\geq} 8+2\sum_{cyc}\frac{a^2}{b^2+c^2}}$$

$$\text{Also, } \frac{R+\sqrt{3}s}{r} \overset{\text{Mitrinovic}}{\geq} \frac{R}{r}+9 \overset{?}{\geq} 8+2\sum_{cyc}\frac{a}{b+c}$$

$$\Leftrightarrow \frac{R}{r}+1 \overset{?}{\geq} 2\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{b+c}\right)-6$$

$$\Leftrightarrow \frac{R}{r}+7 \overset{?}{\geq} \frac{4s}{2s(s^2+2Rr+r^2)}\cdot\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)$$

$$\Leftrightarrow \frac{R+7r}{r} \overset{?}{\geq} \frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2} \Leftrightarrow (R-3r)s^2+r(2R^2+7Rr+5r^2) \underset{(****)}{\overset{?}{\geq}} 0$$

$\boxed{Case\ 1}$ $R-3r \geq 0$ *and* **then : LHS of** $(****) \geq r(2R^2+7Rr+5r^2) > 0$

$\Rightarrow (****)$ **is true (strict inequality)**

$\boxed{Case\ 2}$ $R-3r < 0$ *and* **then : LHS of** $(****)$

$$= -(3r-R)s^2+r(2R^2+7Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\geq} -(3r-R)(4R^2+4Rr+3r^2)+r(2R^2+7Rr+5r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2t^3-3t^2-t-2 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(2t^2+t+1) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (****) \text{ is true } \therefore \boxed{\frac{R+\sqrt{3}s}{r} \overset{(\blacksquare\blacksquare)}{\geq} 8+2\sum_{cyc}\frac{a}{b+c}}$$

$$\therefore (\blacksquare),(\blacksquare\blacksquare) \Rightarrow \frac{R+\sqrt{3}s}{r} \geq 8+2.max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\} \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \min\left\{\sum_{cyc}\sqrt{\frac{n_a}{n_b}},\sum_{cyc}\sqrt{\frac{w_a}{w_b}}\right\}+\frac{R+\sqrt{3}s}{r} \geq 11+2.max\left\{\sum_{cyc}\frac{a^2}{b^2+c^2},\sum_{cyc}\frac{a}{b+c}\right\}$$

$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

We show that:

$$\sum\frac{a^2}{b^2+c^2} \geq \sum\frac{a}{b+c}$$

We have, $\frac{a^2}{b^2+c^2}-\frac{a}{b+c}=\frac{ab(a-b)+ac(a-c)}{(b^2+c^2)(b+c)}$

$$\frac{b^2}{c^2+a^2}-\frac{b}{c+a}=\frac{bc(b-c)+ab(b-a)}{(c^2+a^2)(b+c)}$$

$$\frac{c^2}{a^2+b^2}-\frac{c}{a+b}=\frac{ac(c-a)+bc(c-b)}{(b^2+a^2)(b+a)}$$

Now we obtain,

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}-\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=\sum\frac{ab(a-b)+ac(b-c)}{(b^2+c^2)(b+c)}$$

$$=(a^2+b^2+c^2+ab+bc+ca)\sum\frac{ab(a-b)^2}{(b+c)(c+a)(b^2+c^2)(c^2+a^2)} \geq 0$$

$$\therefore \sum\frac{a^2}{b^2+c^2} \geq \sum\frac{a}{b+c}$$

$$\therefore \max\left\{\sum\frac{a^2}{b^2+c^2};\sum\frac{a}{b+c}\right\} \leq \sum\frac{a^2}{b^2+c^2}$$

$\sum\sqrt{\frac{r_a}{r_b}} \geq 3$ (AM-GM), $\sum\sqrt{\frac{w_a}{w_b}} \geq 3$ (AM-GM)

$$\therefore \min\left\{\sum\sqrt{\frac{r_a}{r_b}},\sum\sqrt{\frac{w_a}{w_b}}\right\} \geq 3$$

We need to show

$$3+\frac{R+\sqrt{3}s}{r}\ge 11+2\sum\frac{a^2}{b^2+c^2} \text{ or } \frac{R+\sqrt{3}s}{r}\ge 8+2\sum\frac{a^2}{b^2+c^2}$$

$$\text{now } \frac{R+\sqrt{3}s}{r}\ge\frac{R+\sqrt{3}\cdot 3\sqrt{3}r}{r}=\frac{R}{r}+9 \quad (1)$$

$$(\because s^2\ge 27r^2)$$

$$\sum\frac{a^2}{b^2+c^2}=\sum_{cyc}\left(\frac{a^2}{b^2+c^2}-\frac{1}{2}\right)+\frac{3}{2}=\sum\frac{2a^2-b^2-c^2}{2(b^2+c^2)}+\frac{3}{2}$$

$$=\sum_{cyc}\frac{a^2-b^2}{2(b^2+c^2)}+\sum_{cyc}\frac{a^2-c^2}{2(b^2+c^2)}+\frac{3}{2}$$

$$=\sum(a^2-b^2)\left(\frac{1}{2(b^2+c^2)}-\frac{1}{2(a^2+c^2)}\right)+\frac{3}{2}$$

$$=\sum\frac{(a-b)^2(a+b)^2}{2(a^2+c^2)(b^2+c^2)}+\frac{3}{2}$$

According Cauchy – Schwarz,

$$(a^2+c^2)(c^2+b^2)\ge c^2(a+b)^2$$

By AM-GM

$$c^2=\frac{(c+a-b+c+b-a)^2}{4}\ge(c+a-b)(c+b-a)$$

$$\therefore (a^2+c^2)(c^2+b^2)\ge(c+a-b)(c+b-a)(a+b)^2$$

$$\therefore \sum\frac{a^2}{b^2+c^2}=\sum\frac{(a-b)^2(a+b)^2}{2(a^2+c^2)(b^2+c^2)}+\frac{3}{2}\le\sum\frac{(a-b)^2}{2(c+a-b)(c+a-b)}+\frac{3}{2}$$

$$=\frac{\sum(a+b-c)\,(a-b)^2}{2(a+b-c)(b+c-a)(c+a-b)}+\frac{3}{2}=\frac{abc}{(a+b-c)(b+c-a)(c+a-b)}+\frac{1}{2}$$

$$=\frac{R}{2r}+\frac{1}{2}$$

$$\therefore 2\sum\frac{a^2}{b^2+c^2}\le\frac{R}{r}+1$$

$$\therefore 8+2\sum\frac{a^2}{b^2+c^2}\le 8+\frac{R}{r}+1=9+\frac{R}{r}\le\frac{R+\sqrt{3}s}{r}$$

(using (1))

1420. In ΔABC the following relationship holds:

$$5-\frac{4r}{R}\le\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\le\frac{2R}{r}-1$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=\frac{a^3+b^3+c^3}{abc}=\frac{2(s^3-3r^2S-6Rrs)}{4Rrs}=\frac{s^2-3r^2-6Rr}{2Rr}$$

$$\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2-3r^2-6Rr}{2Rr}=\frac{4R^2+4Rr-6Rr}{2Rr}=\frac{4R^2-2Rr}{2Rr}$$

$$=\frac{2R}{r}-1$$

Again,

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=\frac{s^2-3r^2-6Rr}{2Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2-3r^2-6Rr}{2Rr}$$

$$=\frac{10Rr-8r^2}{2Rr}=5-\frac{4r}{R}$$

1421. In any Δ ABC, and $\forall$ n $\ge$ 2, the following relationship holds :

$$\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}+\left(\frac{R}{2r}\right)^n\ge 1+\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z \therefore s=x+y+z\Rightarrow a=y+z,$$
$$b=z+x \text{ and } c=x+y$$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(1)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\text{ and } 1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(2)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Now, }\sum_{cyc}\frac{b}{a}=\sum\frac{z+x}{y+z}\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(3)}{=}\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\therefore (1),(2),(3)\Rightarrow\frac{s^2}{r^2}$$

$$\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)^2(y+z)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\overset{(i)}{\geq}xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2$$

Now $\forall\, u,v,w>0, u^3+u^3+v^3\overset{A-G}{\geq}3u^2v, v^3+v^3+w^3\overset{A-G}{\geq}3v^2w$ ***and***
$w^3+w^3+u^3\overset{A-G}{\geq}3w^2u$ ∴ **summing up** : $\sum_{cyc}u^3\geq\sum_{cyc}u^2v$ ***and*** **choosing** $u=xy$,

$$v=yz \text{ and } w=zx, \sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}xy^2\right) \text{ and } \sum_{cyc}x^4y^2\geq 3x^2y^2z^2$$

$$\therefore\sum_{cyc}x^4y^2+\sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}xy^2\right)+3x^2y^2z^2\Rightarrow(i)\text{ is true}$$

$$\Rightarrow\frac{s^2}{r^2}\geq\left(\sum_{cyc}\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Rightarrow\sum_{cyc}\frac{b}{a}\overset{(\blacklozenge)}{\leq}\frac{s^2}{r(4R+r)}$$

Also, $\sum_{cyc}\frac{a}{b}=\sum_{cyc}\frac{y+z}{z+x}\Rightarrow\sum_{cyc}\frac{a}{b}\overset{(4)}{=}\frac{\sum_{cyc}(x+y)(y+z)^2}{\prod_{cyc}(y+z)}\therefore(1),(2),(4)\Rightarrow\frac{s^2}{r^2}$

$$\geq\left(\sum_{cyc}\frac{a}{b}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\geq\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}(x+y)(y+z)^2}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\geq\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}(x+y)(y+z)^2\right)$$

$$\Leftrightarrow\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\overset{(ii)}{\geq}xyz\left(\sum_{cyc}x^2y\right)+3x^2y^2z^2$$

Now $\forall\, u,v,w>0, v^3+v^3+u^3\overset{A-G}{\geq}3v^2u, w^3+w^3+v^3\overset{A-G}{\geq}3w^2v$ ***and***
$u^3+u^3+w^3\overset{A-G}{\geq}3u^2w$ ∴ **summing up** : $\sum_{cyc}u^3\geq\sum_{cyc}uv^2$ ***and*** **choosing** $u=xy$,

$$v=yz \text{ and } w=zx, \sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}x^2y\right) \text{ and } \sum_{cyc}x^2y^4\geq 3x^2y^2z^2$$

$$\therefore\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\geq xyz\left(\sum_{cyc}x^2y\right)+3x^2y^2z^2\Rightarrow(ii)\text{ is true}$$

$$\Rightarrow \frac{s^2}{r^2} \geq \left(\sum_{cyc} \frac{a}{b}\right)\left(1+\frac{4R}{r}\right) \Rightarrow \sum_{cyc} \frac{a}{b} \overset{(\blacklozenge\blacklozenge)}{\leq} \frac{s^2}{r(4R+r)}$$

$$\sum_{cyc} \sqrt[3]{\frac{a}{2b+3c}} = \frac{1}{\sqrt[3]{5}} \cdot \sum_{cyc} \sqrt[3]{\frac{5a}{2b+3c} \cdot 1.1} \overset{G-H}{\geq} \frac{1}{\sqrt[3]{5}} \cdot \sum_{cyc} \frac{\frac{15a}{2b+3c}}{\frac{5a}{2b+3c}+\frac{5a}{2b+3c}+1}$$

$$= \frac{15}{\sqrt[3]{5}} \cdot \sum_{cyc} \frac{a^2}{10a^2+2ab+3ca} \overset{Bergstrom}{\geq} \frac{1}{\sqrt[3]{5}} \cdot \frac{15\left(\sum_{cyc} a\right)^2}{10\sum_{cyc} a^2 + 5\sum_{cyc} ab}$$

$$= \frac{1}{\sqrt[3]{5}} \cdot \frac{15\left(\sum_{cyc} a\right)^2}{10\sum_{cyc} a^2 + 5\sum_{cyc} a^2} = \frac{1}{\sqrt[3]{5}} \cdot \frac{4s^2}{\sum_{cyc} a^2} \Rightarrow \boxed{\sum_{cyc} \sqrt[3]{\frac{a}{2b+3c}} \overset{(\bullet)}{\geq} \frac{1}{\sqrt[3]{5}} \cdot \frac{2s^2}{s^2-4Rr-r^2}}$$

$$\sum_{cyc} \sqrt[3]{\frac{a}{2c+3b}} = \frac{1}{\sqrt[3]{5}} \cdot \sum_{cyc} \sqrt[3]{\frac{5a}{3b+2c} \cdot 1.1} \overset{A-G}{\leq} \frac{1}{\sqrt[3]{5}} \cdot \sum_{cyc} \frac{\frac{5a}{3b+2c}+2}{3}$$

$$= \frac{1}{\sqrt[3]{5}} \cdot \left(\frac{5}{3}\sum_{cyc} \frac{a}{3b+2c}+2\right) \overset{Weighted\ A-H}{\leq} \frac{1}{\sqrt[3]{5}} \cdot \left(\frac{5}{3}\sum_{cyc} \frac{a}{\left(\frac{25}{\frac{3}{b}+\frac{2}{c}}\right)}+2\right)$$

$$= \frac{1}{\sqrt[3]{5}} \cdot \left(\frac{1}{15}\sum_{cyc} \frac{3ca+2ab}{bc}+2\right) = \frac{1}{\sqrt[3]{5}} \cdot \left(\frac{1}{5}\sum_{cyc} \frac{a}{b} + \frac{2}{15}\sum_{cyc} \frac{a}{c}+2\right) \overset{via\ (\blacklozenge),(\blacklozenge\blacklozenge)}{\leq}$$

$$\frac{1}{\sqrt[3]{5}} \cdot \left(\frac{s^2}{3r(4R+r)}+2\right) \Rightarrow \boxed{\sum_{cyc} \sqrt[3]{\frac{a}{2c+3b}} \overset{(\bullet\bullet)}{\leq} \frac{1}{\sqrt[3]{5}} \cdot \left(\frac{s^2+24Rr+6r^2}{3r(4R+r)}\right)}$$

$$\therefore \sum_{cyc} \sqrt[3]{\frac{a}{2b+3c}} + \frac{R^2-4r^2}{4r^2} - \sum_{cyc} \sqrt[3]{\frac{a}{2c+3b}} \overset{via\ (\bullet).(\bullet\bullet)}{\geq}$$

$$\frac{1}{\sqrt[3]{5}} \cdot \left(\frac{2s^2}{s^2-4Rr-r^2} - \frac{s^2+24Rr+6r^2}{3r(4R+r)}\right) + \frac{R^2-4r^2}{4r^2} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{R^2-4r^2}{4r^2} \overset{?}{\geq} \frac{1}{\sqrt[3]{5}} \cdot \left(\frac{s^2+24Rr+6r^2}{3r(4R+r)} - \frac{2s^2}{s^2-4Rr-r^2}\right)$$

$$\Leftrightarrow \frac{R^2-4r^2}{4r^2} \underset{(*)}{\overset{?}{\geq}} \frac{1}{\sqrt[3]{5}} \cdot \frac{(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2}{3r(4R+r)(s^2-4Rr-r^2)}$$

Now, $(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2$

$$= s^4-(4Rr+r^2)s^2-6(4Rr+r^2)^2 \overset{Gerretsen+Euler}{\geq} 3(4Rr+r^2)s^2-(4Rr+r^2)s^2$$

$$-6(4Rr+r^2)^2 = 2(4Rr+r^2)(s^2-12Rr-3r^2) \overset{Gerretsen+Euler}{\geq} 0$$

$$\therefore \frac{1}{\sqrt[3]{5}} \cdot \frac{(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2}{3r(4R+r)(s^2-4Rr-r^2)}$$

$$\overset{\sqrt[3]{5}>\frac{5}{3}}{\leq}\frac{3}{5}\cdot\frac{(s^2-4Rr-r^2)(s^2+24Rr+6r^2)-6r(4R+r)s^2}{3r(4R+r)(s^2-4Rr-r^2)}\overset{?}{\leq}\frac{R^2-4r^2}{4r^2}$$

$$\Leftrightarrow -4rs^4+(20R^3+5R^2r-64Rr^2-16r^3)s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\underset{(**)}{\overset{?}{\geq}}0$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\geq}\begin{pmatrix}-4r(4R^2+4Rr+3r^2)+20R^3\\+5R^2r-64Rr^2-16r^3\end{pmatrix}s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow (20R^3-11R^2r-80Rr^2-28r^3)s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\underset{(***)}{\overset{?}{\geq}}0$$

Case 1 $20R^3-11R^2r-80Rr^2-28r^3\geq 0$ *and* then, LHS of $(***)$

$$\overset{\text{Gerretsen}}{\geq}(20R^3-11R^2r-80Rr^2-28r^3)(16Rr-5r^2)$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow 120t^4-158t^3-263t^2+152t+92\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(120t^2+322t+545)+1044\right)$$

$$\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(***)\text{ is true}$$

Case 2 $20R^3-11R^2r-80Rr^2-28r^3<0$ *and* then, LHS of $(***)$

$$=-\left(-(20R^3-11R^2r-80Rr^2-28r^3)\right)s^2$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)$$

$$\overset{\text{Gerretsen}}{\geq}-\left(-(20R^3-11R^2r-80Rr^2-28r^3)\right)(4R^2+4Rr+3r^2)$$

$$-r(80R^4+40R^3r-699R^2r^2-352Rr^3-44r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow 40t^5-22t^4-172t^3+117t^2-20\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)(40t^4+30t^3+28t^2(t-2)+5t+10)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow (***)$ is true $\therefore$ combining *cases* 1 *and* 2, $(***)\Rightarrow(**)\Rightarrow(*)$ is true $\forall\,\Delta\,ABC$

$$\therefore\frac{R^2-4r^2}{4r^2}\overset{(\blacksquare)}{\geq}\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}-\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}\ \textit{and}\ (n-2).\ln\left(\frac{R}{2r}\right)\overset{\text{Euler}}{\geq}0$$

$$\Rightarrow \ln\left(\frac{R}{2r}\right)^n\geq\ln\left(\frac{R}{2r}\right)^2\Rightarrow\left(\frac{R}{2r}\right)^n-1\geq\frac{R^2-4r^2}{4r^2}\overset{\text{via }(\blacksquare)}{\geq}$$

$$\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}-\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}\Rightarrow\sum_{cyc}\sqrt[3]{\frac{a}{2b+3c}}+\left(\frac{R}{2r}\right)^n$$

$$\geq 1+\sum_{cyc}\sqrt[3]{\frac{a}{2c+3b}}$$ in *any* $\Delta\,ABC$ *and* $\forall\,n\geq 2$, "=" iff $\Delta\,ABC$ is equi*lateral* (QED)

1422. In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^n+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^n\ge 2, n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Adrian Popa – Romania

$$\frac{w_aw_bw_c}{r_ar_br_c}\ge\frac{h_ah_bh_c}{r_ar_br_c}=\frac{\frac{2s}{a}\cdot\frac{2s}{b}\cdot\frac{2s}{c}}{\frac{s}{p-a}\cdot\frac{s}{p-b}\cdot\frac{s}{p-c}}=$$

$$=\frac{8(s-a)(s-b)(s-c)}{abc}=\frac{8p(s-a)(s-b)(s-c)}{p\cdot abc}=$$

$$=\frac{8s^2}{p\cdot 4Rs}=\frac{8\cdot r\cdot p}{p\cdot 4R}=\frac{2r}{R}\Rightarrow$$

$$\Rightarrow\left(\frac{R}{2r}\right)^n+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^n\ge\left(\frac{R}{2r}\right)^n+\left(\frac{2r}{R}\right)^n\overset{MA\ge MG}{\ge}2\sqrt{\left(\frac{R}{2r}\right)^n\cdot\left(\frac{2r}{R}\right)^n}=2$$

Solution 2 by Tapas Das – India

$$\frac{R}{2r}=\frac{abc}{(a+b-c)(b+c-a)(c+a-b)}=\frac{abc}{8(s-a)(s-b)(s-c)}$$

$$\frac{w_aw_bw_c}{r_ar_br_c}\ge\frac{h_ah_bh_c}{r_ar_br_c}=\frac{8F^3}{abcs^2r}=\frac{8F\cdot s(s-a)(s-b)(s-c)}{abc\cdot s^2\cdot r}$$

$$=\frac{8F(s-a)(s-b)(s-c)}{abc\cdot F}=\frac{8(s-a)(s-b)(s-c)}{abc}$$

$$\therefore\left(\frac{R}{2r}\right)^n+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^n\ge$$

$$\ge\left[\frac{abc}{8(s-a)(s-b)(s-c)}\right]^2+\left[\frac{8(s-a)(s-b)(s-c)}{abc}\right]^n$$

$$\overset{AM-GM}{\ge}2\sqrt{\left[\frac{abc}{8(s-a)(s-b)(s-c)}\cdot\frac{8(s-a)(s-b)(s-c)}{abc}\right]^n}=2$$

1423. In any Δ ABC, the following relationships holds :

$$\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{n_a}{n_b}}\right)\geq\sum_{cyc}\sqrt{\frac{n_b}{n_a}};\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{g_a}{g_b}}\right)\geq\sum_{cyc}\frac{g_b}{g_a};$$

and in acute Δ ABC with $a=\min\{a,b,c\}$, holds : $g_a+h_a>n_a$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Tapas Das-India

$$\frac{n_a}{n_b}+\frac{n_b}{n_c}+\frac{n_c}{n_a}\leq\sqrt{\left(\sum n_a^2\right)\left(\sum\frac{1}{n_a^2}\right)}\quad\textbf{(1)}$$

Since $n_a\geq h_a$ (analog)

$$\therefore\frac{1}{n_a^2}+\frac{1}{n_b^2}+\frac{1}{n_c^2}\leq\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}=\frac{\sum a^2}{(2F)^2}\overset{Leibnitz's}{\leq}\left(\frac{3R}{2F}\right)^2$$

$$n_a^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-s\cdot\frac{a^2-(b-c)^2}{a}$$

$$=s^2-\frac{4s(s-b)(s-c)}{a}=s^2-\frac{4s^2r^2}{a(s-a)}$$

$$=s^2-4sr^2\left(\frac{1}{s-a}+\frac{1}{a}\right)=s^2-4rr_a-2rh_a\ \text{(analog)}$$

$$\therefore\sum n_a^2=3s^2-4r\left(\sum r_a\right)-2r\left(\sum h_a\right)$$

$$=3s^2-4r(4R+r)-2r\frac{s^2+r(4R+r)}{2R}=\frac{(3R-r)s^2-r(4R+r)^2}{R}$$

By Doucets, $(4R+r)^2\geq 3s^2$ we get

$$\sum n_a^2\leq\frac{(3R-r)s^2-3rs^2}{R}=\left(3-\frac{4r}{R}\right)s^2\overset{AM-GM}{\leq}\left(\frac{R}{2r}\right)^2\cdot s^2=\left(\frac{Rs}{2r}\right)^2$$

$$\therefore\sum\frac{n_a}{n_b}\leq\frac{Rs}{2r}\cdot\frac{3R}{2F}=3\left(\frac{R}{2r}\right)^2$$

$$\text{Now }\sqrt{\frac{n_a}{n_b}}\overset{CBS}{\leq}\sqrt{3\left(\sum\frac{n_a}{n_b}\right)}\leq\sqrt{3\cdot 3\cdot\left(\frac{R}{2r}\right)^2}=\frac{3R}{2r}$$

$$\therefore\frac{R}{2r}\left(\sum\sqrt{\frac{n_a}{n_b}}\right)\geq\frac{3R}{2r}\text{ (AM-GM) [Since }\sqrt{\frac{n_a}{n_b}}\geq 3]$$

$$\therefore \frac{R}{2r}\left(\sum\sqrt{\frac{n_a}{n_b}}\right) \geq \sqrt{\frac{n_a}{n_b}}$$

$$\sum\frac{g_b}{g_a} \leq \sqrt{\left(\sum g_b^2\right)\left(\sum\frac{1}{g_a^2}\right)} \leq \sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{r_a^2}\right)}$$

$$= \sqrt{s^2 \cdot \frac{\sum a^2}{4F^2}} \overset{Leibnitz}{\leq} \sqrt{\frac{9R^2}{4r^2}}$$

Now $\frac{R^2}{4r^2}\left(\sum\sqrt{\frac{g_a}{g_b}}\right) \geq \frac{R^2}{4r^2} \cdot 3$ (AM-GM)

We need to show

$\frac{3R^2}{4r^2} \geq \frac{3R}{2r}$ or $\frac{R}{2r} \geq 1 \therefore R \geq 2r$ (Euler)

Note: $9a^2 = s(s-a) - \frac{(b-c)^2(s-a)}{a} \leq s(s-a)$ (analog)

If we take: $a = c = 6$ and $b = 2$

We have $s = \frac{6+6+2}{2} = 7$

$$n_a^2 = \frac{b^2(s-c) + c^2(s-b) - a(s-c)(s-b)}{a}$$

$$= \frac{4(7-6)+36(7-2)-6(7-6)(7-2)}{6} = \frac{77}{3} \therefore n_a = \sqrt{\frac{77}{3}} = 5 \cdot 06 \text{ (APP)}$$

$$h_a^2 = \frac{4F^2}{a^2} = \frac{4s(s-a)(s-b)(s-c)}{a^2} = \frac{4\times 7\times 1\times 1\times 5}{9} = \frac{35}{9}$$

$$\therefore h_a = \sqrt{\frac{35}{9}} = 1 \cdot 97 \text{ (APP)}$$

$$g_a^2 = s(s-a) - \frac{(s-a)(b-c)^2}{a}$$

$$= 7 - \frac{16}{6} = \frac{13}{3} \therefore g_a = \sqrt{\frac{13}{3}} = 2 \cdot 08$$

$$\therefore g_a + h_a = 2.08 + 1.97 = 4.05 < n_a$$

$$(\because n_a = 5.06)$$

$\therefore$ so 3rd problem-Not always true

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2 + a(s-b)(s-c)$$

$$\text{and } b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2 + a(s-b)(s-c) \text{ and } (m)+(n) \Rightarrow$$

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$

$$= 2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2) = 2(n_a^2+g_a^2) + a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2$$

$$= 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow (b-c)^2 + 4s(s-a) + (b-c)^2$$

$$= 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 \overset{(1)}{=} (b-c)^2 + 2s(s-a)$$

$$\text{Now, } \sum_{cyc}\frac{n_a}{R} = \sum_{cyc}\frac{2n_ah_a}{bc} \le \sum_{cyc}\frac{2n_ag_a}{bc} \overset{A-G}{\le} \sum_{cyc}\frac{n_a^2+g_a^2}{bc} \overset{\text{via }(1)}{=} \frac{(b-c)^2+2s(s-a)}{bc}$$

$$= \frac{1}{4Rrs}\left(\sum_{cyc}\left(a(b^2+c^2-2bc)\right) + 2s\sum_{cyc}\left(a(s-a)\right)\right)$$

$$= \frac{1}{4Rrs}\left(2s(s^2+4Rr+r^2) - 36Rrs + 2s\left(2s^2 - 2(s^2-4Rr-r^2)\right)\right)$$

$$= \frac{s^2-6Rr+3r^2}{2Rr} \Rightarrow 2r\sum_{cyc} n_a \overset{(*)}{\le} s^2-6Rr+3r^2$$

$$\sum_{cyc}\sqrt{\frac{n_b}{n_a}} \overset{CBS}{\le} \sqrt{\sum_{cyc} n_a} \cdot \sqrt{\sum_{cyc}\frac{1}{n_a}} \le \sqrt{\sum_{cyc} n_a} \cdot \sqrt{\sum_{cyc}\frac{1}{h_a}} = \sqrt{\frac{\sum_{cyc} n_a}{r}} \overset{?}{\le} \frac{3R}{2r}$$

$$\Leftrightarrow \frac{\sum_{cyc} n_a}{r} \overset{?}{\le} \frac{9R^2}{4r^2} \Leftrightarrow 2r\sum_{cyc} n_a \underset{(\bullet)}{\overset{?}{\le}} \frac{9R^2}{2} \text{ and via } (*),$$

$$2r\sum_{cyc} n_a \le s^2-6Rr+3r^2 \overset{?}{\le} \frac{9R^2}{2} \Leftrightarrow 2s^2 \underset{(\bullet\bullet)}{\overset{?}{\le}} 9R^2+12Rr-6r^2 \text{ and via Gerretsen,}$$

$$2s^2 \overset{\text{Gerretsen}}{\le} 8R^2+8Rr+6r^2 \overset{?}{\le} 9R^2+12Rr-6r^2 \Leftrightarrow R^2+4Rr-12r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R-2r)(R+6r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\Rightarrow \sum_{cyc}\sqrt{\frac{n_b}{n_a}} \le \frac{3R}{2r} \overset{A-G}{\le} \frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{n_a}{n_b}}\right) \therefore \boxed{\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{n_a}{n_b}}\right) \ge \sum_{cyc}\sqrt{\frac{n_b}{n_a}}}$$

$$\text{Again, } AI^2 = bc - 4Rr \Leftrightarrow \left(\frac{r}{\left(\frac{r}{4R}\right)}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2$$

$$= 16R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 16R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow \sin\frac{B}{2}\sin\frac{C}{2}$$

$$= \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{A}{2} \Leftrightarrow \cos\frac{B+C}{2} = \sin\frac{A}{2} \to \text{true}$$

$$\therefore AI^2 = bc - 4Rr \text{ and analogs } \therefore \sum_{cyc}\frac{g_b}{g_a} \overset{\text{Triangle-inequality}}{\le} \sum_{cyc}\frac{AI+r}{h_a}$$

$$=\sum_{cyc}\frac{a.AI+r.a}{2rs}\overset{CBS}{\leq}\frac{1}{2rs}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}AI^2}+\frac{2rs}{2rs}\overset{Leibnitz}{\leq}\frac{3R}{2rs}.\sqrt{\sum_{cyc}bc-12Rr}+1$$

$$=\frac{3R.\sqrt{s^2-8Rr+r^2}}{2rs}+1\overset{?}{\leq}\frac{3R^2}{4r^2}\Leftrightarrow\frac{s^2(3R^2-4r^2)^2}{4r^2}\overset{?}{\geq}9R^2(s^2-8Rr+r^2)$$

$$\Leftrightarrow(9R^4-60R^2r^2+16r^4)s^2+r^3(288R^3-36R^2r)\underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}}0$$

$\boxed{Case\ 1}$ $9R^4-60R^2r^2+16r^4\geq 0$ and then, LHS of $(\bullet\bullet\bullet)$

$\geq 270R^3+18R^2(R-2r)\overset{Euler}{\geq}270R^3>0\Rightarrow(\bullet\bullet\bullet)$ is true (strict inequality)

$\boxed{Case\ 2}$ $9R^4-60R^2r^2+16r^4<0$ and then, LHS of $(\bullet\bullet\bullet)$

$$=-s^2.\left(-(9R^4-60R^2r^2+16r^4)\right)+r^3(288R^3-36R^2r)\overset{Gerretsen}{\geq}$$

$$-(4R^2+4Rr+3r^2).\left(-(9R^4-60R^2r^2+16r^4)\right)+r^3(288R^3-36R^2r)\overset{?}{\geq}0$$

$$\Leftrightarrow 36t^6+36t^5-213t^4+48t^3-152t^2+64t+48\overset{?}{\geq}0\quad\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(36t^5+108t^4+3t^3+26t^2+22t(t-2)+6(t-2)(t+2)\right)\overset{?}{\geq}0\to\text{true}$$

$\because t\overset{Euler}{\geq}2\Rightarrow(\bullet\bullet\bullet)$ is true and combining cases 1 and 2, $(\bullet\bullet\bullet)$ is true $\forall$ triangles

$$\Rightarrow\sum_{cyc}\frac{g_b}{g_a}\leq\frac{3R^2}{4r^2}\overset{A-G}{\leq}\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{g_a}{g_b}}\right)\therefore\boxed{\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{g_a}{g_b}}\right)\geq\sum_{cyc}\frac{g_b}{g_a}}$$

We now *shall* prove : $g_a+h_a>n_a$ *in* acute Δ ABC with $a=\min\{a,b,c\}$

$$\frac{b+c}{a}\geq\frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

$$\Leftrightarrow(b+c)(a+b-c).(b+c-a)(c+a-b)\geq 2a^2bc$$

$$\Leftrightarrow(ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)\geq 2a^2bc$$

$$\Leftrightarrow 2a^2b^2+2a^2bc+2ab(b^2-c^2)-(a^2+b^2-c^2)(ab+ac+b^2-c^2)\geq 2a^2bc$$

$$\Leftrightarrow 2a^2b^2-(a^2+b^2-c^2)(ab+ac)+2ab(b^2-c^2)-2ab(b^2-c^2).\cos C\geq 0$$

$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-(b^2-c^2)(ab+ac)+2ab(b^2-c^2).2\sin^2\frac{C}{2}\geq 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)-(b^2-c^2)(ab+ac)+(b^2-c^2)(c^2-(a-b)^2)\geq 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)+(b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac)\geq 0$$

$$\Leftrightarrow\left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$
$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac)\geq 0$$

$$\Leftrightarrow(c^2+a^2-b^2)(2b^2-ab-ac)+(b^2-c^2)(b^2+c^2-a^2-2ac)\geq 0$$

$$\Leftrightarrow(b^2-c^2)(b^2+c^2-2ac)-(b^2-c^2)(2b^2-ab-ac)$$
$$+a^2(2b^2-ab-ac)-a^2(b^2-c^2)\geq 0$$

$$\Leftrightarrow(b^2-c^2)\left((c^2-ca)-(b^2-ab)\right)+a^2\left((c^2-ca)+(b^2-ab)\right)\geq 0$$

$$\Leftrightarrow(c^2-ca)(b^2-c^2+a^2)+(b^2-ab)(a^2+c^2-b^2)\geq 0$$

$$\Leftrightarrow c(c-a)(a^2+b^2-c^2)+b(b-a)(c^2+a^2-b^2)\geq 0$$

$\to$ true $\because\Delta$ ABC being *acute* $\Rightarrow(a^2+b^2-c^2),(c^2+a^2-b^2)>0$ and

$$a=\min\{a,b,c\}\Rightarrow(c-a),(b-a)\geq 0\therefore\frac{b+c}{a}\geq\frac{R}{r}$$

$$\therefore\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}\geq\frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Rightarrow 2\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\cos\frac{B-C}{2}\geq 1\Rightarrow 2\cos^2\frac{B-C}{2}-2\cos\frac{B-C}{2}\sin\frac{A}{2}-1\overset{(\blacksquare)}{\geq}0$$

$$\text{We have } n_a+r_a\overset{CBS}{\leq}\sqrt{2(n_a^2+r_a^2)}\overset{via\,(*)}{=}\sqrt{2(s^2-2r_ah_a+r_a^2)}\overset{?}{\leq}2h_a$$

$$\Leftrightarrow s^2-2r_ah_a+s^2\tan^2\frac{A}{2}\overset{?}{\leq}2h_a^2$$

$$\Leftrightarrow\frac{8s^2.16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}+\frac{4.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}s.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\overset{?}{\geq}s^2\sec^2\frac{A}{2}$$

$$\Leftrightarrow 8\sin^2\frac{B}{2}\sin^2\frac{C}{2}+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\overset{?}{\geq}1$$

$$\Leftrightarrow 2\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)^2+2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\overset{?}{\geq}1$$

$$\left(\because 2\sin\frac{B}{2}\sin\frac{C}{2}=\cos\frac{B-C}{2}-\cos\frac{B+C}{2}=\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)$$

$$\Leftrightarrow 2\left(\cos^2\frac{B-C}{2}+\sin^2\frac{A}{2}-2\sin\frac{A}{2}.\cos\frac{B-C}{2}\right)+2\sin\frac{A}{2}.\cos\frac{B-C}{2}$$

$$-2\sin^2\frac{A}{2}-1\overset{?}{\geq}0\Leftrightarrow 2\cos^2\frac{B-C}{2}-2\cos\frac{B-C}{2}\sin\frac{A}{2}-1\overset{?}{\geq}0\to\text{true via }(\blacksquare)$$

$$\therefore n_a+r_a\leq 2h_a\leq h_a+g_a\Rightarrow g_a+h_a\geq n_a+r_a>n_a$$

$$\therefore\boxed{\text{in acute }\Delta ABC\text{ with }a=\min\{a,b,c\},\text{holds}: g_a+h_a>n_a}\ (QED)$$

1424.

In any Δ ABC and ∀ n ∈ ℕ, the following relationship holds :

$$\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}.r^n}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{WLOG assuming } a\geq b\geq c\Rightarrow w_a^x\leq w_b^x\leq w_c^x\text{ and}$$

$$\frac{w_a^2}{(w_b+w_c)^2}\leq\frac{w_b^2}{(w_c+w_a)^2}\leq\frac{w_c^2}{(w_a+w_b)^2}\text{ and}$$

$$\frac{1}{(w_b+w_c)^2}\leq\frac{1}{(w_c+w_a)^2}\leq\frac{1}{(w_a+w_b)^2}\text{ where } x>0\to(1)$$

$$w_aw_bw_c\geq h_ah_bh_c=\frac{2r^2s^2}{R}\overset{Gerretsen}{\geq}\frac{r^2(27Rr+5r(R-2r))}{R}\overset{Euler}{\geq}\frac{r^2(27Rr)}{R}$$

$$\therefore w_aw_bw_c\geq 27r^3\to(2)$$

$$\sum_{cyc} w_a \geq \sum_{cyc} h_a = 2rs\sum_{cyc}\frac{1}{a} \overset{Bergstrom}{\geq} \frac{2rs.9}{2s} \therefore \sum_{cyc} w_a \geq 9r \to (3)$$

Case 1 $n = 1$ *and then*, $\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}$

$$= \sum_{cyc}\frac{w_a(w_a^2+w_bw_c)}{(w_b+w_c)^2} \overset{A-G}{\geq} \sum_{cyc}\frac{2w_a^2.\sqrt{w_bw_c}}{(w_b+w_c)^2} = 2\sum_{cyc}\frac{\left(\frac{w_a}{w_b+w_c}\right)^2}{\frac{1}{\sqrt{w_bw_c}}} \overset{Bergstrom}{\geq}$$

$$\frac{2\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^2}{\sum_{cyc}\frac{1}{\sqrt{w_bw_c}}} \overset{\substack{Nesbitt\\and\\CBS}}{\geq} \frac{\frac{2.9}{4}}{\sqrt{\sum_{cyc}\frac{1}{w_b}}.\sqrt{\sum_{cyc}\frac{1}{w_c}}} \overset{w_a \geq h_a \text{ and analogs}}{\geq} \frac{\frac{9}{2}}{\sum_{cyc}\frac{1}{h_a}} = \frac{9r}{2}$$

$$\therefore \frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2} \geq \frac{3^{n+1}.r^n}{2}$$

Case 2 $n = 2$ *and then*, $\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}$

$$= \sum_{cyc}\frac{w_a^2(w_a^2+w_bw_c)}{(w_b+w_c)^2} \overset{A-G}{\geq} \sum_{cyc}\frac{2w_a^3.\sqrt{w_bw_c}}{(w_b+w_c)^2} \overset{G-H}{\geq} \sum_{cyc}\frac{4w_a^3w_bw_c}{(w_b+w_c)^3} = 4\sum_{cyc}\frac{\left(\frac{w_a}{w_b+w_c}\right)^3}{\frac{1}{w_bw_c}}$$

$$\overset{Holder}{\geq} \frac{4\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^3}{3\sum_{cyc}\frac{1}{w_bw_c}} \overset{Nesbitt}{\geq} \frac{\frac{4.27}{8}.w_aw_bw_c}{3\sum_{cyc}w_a} \geq \frac{9.\frac{16Rr^2s^2}{s^2+2Rr+r^2}}{2.\sqrt{3}s} \overset{?}{\geq} \frac{27r^2}{2}$$

$$\Leftrightarrow 256R^2s^2 \underbrace{\overset{?}{\geq}}_{(*)} 27(s^2+2Rr+r^2)^2 \text{ and LHS of } (*) \overset{Mitrinovic}{\geq} \frac{256.4s^4}{27}$$

$$\overset{?}{\geq} 27(s^2+2Rr+r^2)^2 \Leftrightarrow 32s^2 \overset{?}{\geq} 27(s^2+2Rr+r^2) \Leftrightarrow 5s^2 \overset{?}{\geq} 27(2Rr+r^2)$$

$$\text{and } 5s^2 \overset{Gerretsen}{\geq} \frac{5}{2}(27Rr+5r(R-2r)) \overset{Euler}{\geq} \frac{5}{2}(27Rr) \overset{?}{\geq} 27(2Rr+r^2) \Leftrightarrow R \overset{?}{\geq} 2r$$

$$\to \text{true via Euler} \therefore \sum_{cyc}\frac{w_a^2(w_a^2+w_bw_c)}{(w_b+w_c)^2} \geq \frac{27r^2}{2}$$

$$\therefore \frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2} \geq \frac{3^{n+1}.r^n}{2}$$

Case 3 $n = 3$ *and then*, $\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}$

$$= \sum_{cyc}\frac{w_a^3(w_a^2+w_bw_c)}{(w_b+w_c)^2} \overset{A-G}{\geq} \frac{2w_a^4.\sqrt{w_bw_c}}{(w_b+w_c)^2} \overset{G-H}{\geq} \sum_{cyc}\frac{4w_a^4w_bw_c}{(w_b+w_c)^3}$$

$$= 4w_aw_bw_c\sum_{cyc}\left(\frac{w_a}{w_b+w_c}\right)^3 \overset{Holder}{\geq} \frac{4}{9}.w_aw_bw_c.\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^3 \overset{\substack{Nesbitt\\and\\(2)}}{\geq} \frac{4}{9}.27r^3.\frac{.27}{8}$$

$$=\frac{81r^3}{2}\therefore\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}.r^n}{2}$$

$\boxed{Case\ 4}$ $n\in\mathbb{N}-\{1,2,3\}$ and then, $\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}$

$$+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}=\sum_{cyc}\left(w_a^n.\left(\frac{w_a}{w_b+w_c}\right)^2\right)+w_aw_bw_c.\sum_{cyc}\frac{w_a^{n-1}}{(w_b+w_c)^2}$$

$$\overset{\text{Chebyshev,via (1) and (2)}}{\geq}\frac{1}{3}\left(\sum_{cyc}w_a^n\right)\left(\sum_{cyc}\left(\frac{w_a}{w_b+w_c}\right)^2\right)$$

$$+9r^3.\left(\sum_{cyc}w_a^{n-1}\right)\left(\sum_{cyc}\frac{1^3}{(w_b+w_c)^2}\right)\overset{\text{Repeated Chebyshev and Radon}}{\geq}\frac{1}{3.3.3^{n-1}}\left(\sum_{cyc}w_a\right)^n\left(\sum_{cyc}\frac{w_a}{w_b+w_c}\right)^2$$

$$+\frac{9r^3}{3^{n-2}}\left(\sum_{cyc}w_a\right)^{n-1}.\frac{27}{4\left(\sum_{cyc}w_a\right)^2}\overset{\text{Nesbitt}}{\geq}\frac{1}{3^{n+1}}.\frac{9}{4}.\left(\sum_{cyc}w_a\right)^n$$

$$+\frac{3^5r^3}{4.3^{n-2}}.\left(\sum_{cyc}w_a\right)^{n-3}\overset{\text{via (3)}}{\geq}\frac{1}{3^{n+1}}.\frac{9}{4}.3^{2n}.r^n+\frac{3^5r^3}{4.3^{n-2}}.3^{2n-6}.r^{n-3}$$

$$=\frac{3^{n+1}.r^n}{4}+\frac{3^{n+1}.r^n}{4}\therefore\sum_{cyc}\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}\geq\frac{3^{n+1}.r^n}{2}$$

$\therefore$ combining *all* cases, in *any* ΔABC *and* $\forall n\in\mathbb{N}$,

$$\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}.r^n}{2},$$

$''=''$ iff ΔABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$\frac{w_a{}^n(w_a{}^2+w_bw_c)}{(w_b+w_c)^2}=\frac{w_a{}^n}{(w_b+w_c)^2}(w_a-w_b)(w_a-w_c)+\frac{w_a{}^{n+1}}{w_b+w_c}\ \text{(and analogs)}$$

WLOG, we may assume that $w_a\geq w_b\geq w_c$.

$$\text{We have }\frac{w_a{}^n}{(w_b+w_c)^2}\geq\frac{w_b{}^n}{(w_c+w_a)^2}\geq\frac{w_c{}^n}{(w_a+w_b)^2},$$

then by the Generalized Schur inequality, we have

$$\sum_{cyc}\frac{w_a{}^n}{(w_b+w_c)^2}(w_a-w_b)(w_a-w_c)\geq 0.$$

Therefore

$$\sum_{cyc}\frac{w_a{}^n(w_a{}^2+w_bw_c)}{(w_b+w_c)^2}\geq\sum_{cyc}\frac{w_a{}^{n+1}}{w_b+w_c}\overset{\text{Hölder}}{\geq}\frac{(w_a+w_b+w_c)^{n+1}}{3^{n-1}.2(w_a+w_b+w_c)}\geq\frac{(h_a+h_b+h_c)^2}{6}$$

$$\overset{AM-HM}{\geq}\frac{1}{6}\left(\frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^2=\frac{(9r)^2}{6}=\frac{27r^2}{2}.$$

as desired. Equality holds if and only if ΔABC is equilateral.

1425. In any Δ ABC and ∀ m, n ∈ ℕ such that : n ≥ m − 2,

the following relationship holds :

$$\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove that $\forall\, x,y,z>0,\ \sum_{cyc}\frac{x^3+xyz}{(y+z)^3}\overset{(\blacklozenge)}{\geq}\frac{3}{4}$

$$\sum_{cyc}\frac{x^3+xyz}{(y+z)^3}=\sum_{cyc}\frac{x^3}{(y+z)^3}+xyz\sum_{cyc}\frac{1^4}{(y+z)^3}\overset{\substack{\text{Chebyshev}\\ \text{and}\\ \text{Radon}}}{\geq}$$

$$\frac{1}{3}\left(\sum_{cyc}\frac{x}{y+z}\right)\left(\sum_{cyc}\frac{x^2}{(y+z)^2}\right)+\frac{81xyz}{8\left(\sum_{cyc}x\right)^3}\overset{\text{Nesbitt}}{\geq}\frac{1}{3}\cdot\frac{3}{2}\cdot\sum_{cyc}\frac{x^4}{x^2y^2+x^2z^2+2x^2yz}$$

$$+\frac{81xyz}{8\left(\sum_{cyc}x\right)^3}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x^2\right)^2}{4\left(\sum_{cyc}x^2y^2+xyz\sum_{cyc}x\right)}+\frac{81xyz}{8\left(\sum_{cyc}x\right)^3}\overset{?}{\geq}\frac{3}{4}$$

$$\Leftrightarrow\boxed{\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}x^2y^2+xyz\sum_{cyc}x}+\frac{81xyz}{2\left(\sum_{cyc}x\right)^3}\overset{?}{\underset{(\blacksquare)}{\geq}}3}$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\rightarrow(1)$

$\Rightarrow x=s-X, y=s-Y, z=s-Z\Rightarrow xyz=r^2s\rightarrow(2)$ and also, such substitutions $\Rightarrow\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)\Rightarrow\sum_{cyc}xy=4Rr+r^2\rightarrow(3)$ and

$$\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via (1) and (3)}}{=}s^2-2(4Rr+r^2)$$

$$\Rightarrow\sum_{cyc}x^2=s^2-8Rr-2r^2\rightarrow(4)\text{ and also, }\sum_{cyc}x^2y^2=\left(\sum_{cyc}xy\right)^2-2xyz\sum_{cyc}x$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s \therefore \sum_{cyc} x^2y^2 = r^2((4R + r)^2 - 2s^2) \to (5)$$

$$\therefore \text{via } (1),(2),(4),(5),(\blacksquare) \Leftrightarrow \frac{(s^2 - 8Rr - 2r^2)^2}{r^2((4R + r)^2 - 2s^2) + r^2 s.s} + \frac{81r^2 s}{2s^3} \ge 3$$

$$\Leftrightarrow 2s^6 - (32Rr + 2r^2)s^4 + r^2s^2(32R^2 + 16Rr - 79r^2)$$

$$+r^4(1296R^2 + 648Rr + 81r^2) \overset{(\blacksquare\blacksquare)}{\ge} 0 \text{ and } \because 2(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0$$

$\therefore$ in order to prove $(\blacksquare\blacksquare)$, it suffices to prove : LHS of $(\blacksquare\blacksquare)$

$$\ge 2(s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (64Rr - 32r^2)s^4 - r^2s^2(1504R^2 - 976Rr + 229r^2)$$

$$+r^3(8192R^3 - 6384R^2r + 3048Rr^2 - 169r^3) \overset{(\blacksquare\blacksquare\blacksquare)}{\ge} 0 \text{ and}$$

$$\because (64Rr - 32r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0, \therefore \text{ in order to prove } (\blacksquare\blacksquare\blacksquare),$$

it suffices to prove : LHS of $(\blacksquare\blacksquare\blacksquare) \ge (64Rr - 32r^2)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (544R^2 - 688Rr + 91r^2)s^2 \ge r(8192R^3 - 12048R^2r + 3672Rr^2 - 631r^3)$$

$$\Leftrightarrow 64t^3 - 210t^2 + 153t + 22 \ge 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)((t-2)(64t + 46) + 81)$$

$$\ge 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\blacksquare\blacksquare\blacksquare) \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{x^3 + xyz}{(y+z)^3} \ge \frac{3}{4}\ \forall\, x,y,z > 0$$

Case 1 $n - m = -2$ and then : $\dfrac{r_a^n(r_a^2 + r_br_c)}{(r_b + r_c)^m} + \dfrac{r_b^n(r_b^2 + r_cr_a)}{(r_c + r_a)^m} + \dfrac{r_c^n(r_c^2 + r_ar_b)}{(r_a + r_b)^m}$

$$\ge \frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}} \Leftrightarrow \boxed{\sum_{cyc} \frac{r_a^{m-2}(r_a^2 + r_br_c)}{(r_b + r_c)^m} \overset{(*)}{\ge} \frac{3}{2^{m-1}}}$$

$$\text{Now}, n - m = -2 \Rightarrow m - 2 = n \ge 1 \Rightarrow m - 3 \ge 0 \therefore \sum_{cyc} \frac{r_a^{m-2}(r_a^2 + r_br_c)}{(r_b + r_c)^m}$$

$$= \sum_{cyc}\left(\left(\frac{r_a}{r_b + r_c}\right)^{m-3} \cdot \left(\frac{r_a^3 + r_ar_br_c}{(r_b + r_c)^3}\right)\right) \overset{\text{Chebyshev}}{\ge} \left(\sum_{cyc}\left(\frac{r_a}{r_b + r_c}\right)^{m-3}\right)\left(\sum_{cyc}\frac{r_a^3 + r_ar_br_c}{(r_b + r_c)^3}\right)$$

$$\left(\begin{array}{c} \because \text{WLOG assuming } a \ge b \ge c \Rightarrow \left(\frac{r_a}{r_b + r_c}\right)^{m-3} \ge \left(\frac{r_b}{r_c + r_a}\right)^{m-3} \left(\frac{r_c}{r_a + r_b}\right)^{m-3} \text{ and} \\ r_a^3 + r_ar_br_c \ge r_b^3 + r_ar_br_c \ge r_c^3 + r_ar_br_c \text{ alongwith } \frac{1}{(r_b + r_c)^3} \ge \frac{1}{(r_c + r_a)^3} \ge \frac{1}{(r_a + r_b)^3} \\ \Rightarrow \frac{r_a^3 + r_ar_br_c}{(r_b + r_c)^3} \ge \frac{r_b^3 + r_ar_br_c}{(r_c + r_a)^3} \ge \frac{r_c^3 + r_ar_br_c}{(r_a + r_b)^3} \end{array}\right)$$

$$\overset{\substack{\text{Repeated Chebyshev} \\ \text{and} \\ \text{via } (\blacklozenge)}}{\ge} \frac{1}{3} \cdot \frac{1}{3^{m-4}} \cdot \left(\sum_{cyc} \frac{r_a}{r_b + r_c}\right)^{m-3} \cdot \frac{3}{4} \overset{\text{Nesbitt}}{\ge} \frac{1}{4.3^{m-4}} \cdot \frac{3^{m-3}}{2^{m-3}} = \frac{3}{2^{m-1}}$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{r_a^n(r_a^2 + r_br_c)}{(r_b + r_c)^m} + \frac{r_b^n(r_b^2 + r_cr_a)}{(r_c + r_a)^m} + \frac{r_c^n(r_c^2 + r_ar_b)}{(r_a + r_b)^m} \ge \frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

Case 2 $n-m=-1$ *and then* : $\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}\Leftrightarrow\boxed{\sum_{cyc}\frac{r_a^{m-1}(r_a^2+r_br_c)}{(r_b+r_c)^m}\overset{(**)}{\geq}\frac{9r}{2^{m-1}}}$$

$$\text{Now, }\sum_{cyc}\frac{r_a^{m-1}(r_a^2+r_br_c)}{(r_b+r_c)^m}\overset{A-G}{\geq}2\sum_{cyc}\frac{r_a^m.\sqrt{r_br_c}}{(r_b+r_c)^m}=2\sum_{cyc}\frac{\left(\frac{r_a}{r_b+r_c}\right)^m}{\frac{1}{\sqrt{r_br_c}}}\overset{Holder}{\geq}$$

$$2.\frac{\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^m}{3^{m-2}.\sum_{cyc}\frac{1}{\sqrt{r_br_c}}}\overset{\substack{Nesbitt\\and\\CBS}}{\geq}\frac{2}{3^{m-2}}.\frac{3^m}{2^m}.\frac{1}{\sqrt{\sum_{cyc}\frac{1}{r_a}}.\sqrt{\sum_{cyc}\frac{1}{r_a}}}=\frac{9r}{2^{m-1}}\Rightarrow(**)\text{ is true}$$

$$\therefore\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

Case 3 $n-m=0$ *and then* : $\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}\Leftrightarrow\boxed{\sum_{cyc}\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^n}\overset{(**)}{\geq}\frac{27r^2}{2^{n-1}}}$$

$$\text{Now, }\sum_{cyc}\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^n}\overset{A-G}{\geq}2\sum_{cyc}\frac{r_a^{n+1}.\sqrt{r_br_c}}{(r_b+r_c)^n}\overset{G-H}{\geq}4r_ar_br_c\sum_{cyc}\left(\left(\frac{r_a}{r_b+r_c}\right)^n.\frac{1}{r_b+r_c}\right)\overset{Chebyshev}{\geq}$$

$$\frac{4rs^2}{3}\left(\sum_{cyc}\left(\frac{r_a}{r_b+r_c}\right)^n\right)\left(\sum_{cyc}\frac{1}{r_b+r_c}\right)\overset{\substack{Repeated\ Chebyshev\\and\\Bergstrom}}{\geq}\frac{4rs^2}{3.3^{n-1}}.\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^n.\frac{9}{2(4R+r)}$$

$$\overset{Nesbitt}{\geq}\frac{2rs^2}{3^n}.\frac{3^n}{2^n}.\frac{9}{4R+r}\overset{?}{\geq}\frac{27r^2}{2^{n-1}}\Leftrightarrow s^2\overset{?}{\geq}3r(4R+r)\to\text{true}$$

$$\because s^2\overset{Gerretsen}{\geq}3r(4R+r)+4r(R-2r)\overset{Euler}{\geq}3r(4R+r)$$

$$\Rightarrow(**)\text{ is true}\therefore\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

Case 4 $n-m\geq1$ *and then* : $\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$

$$\overset{A-G}{\geq}2\sum_{cyc}\frac{r_a^{n+1}.\sqrt{r_br_c}}{(r_b+r_c)^m}\overset{G-H}{\geq}4r_ar_br_c.\sum_{cyc}\frac{r_a^n}{(r_b+r_c)^{m+1}}$$

$$=4rs^2.\sum_{cyc}\left(\left(\frac{r_a}{r_b+r_c}\right)^{m+1}.r_a^{n-m-1}\right)\overset{Chebyshev}{\geq}\frac{4rs^2}{3}\left(\sum_{cyc}\left(\frac{r_a}{r_b+r_c}\right)^{m+1}\right)\left(\sum_{cyc}r_a^{n-m-1}\right)$$

$$\overset{Repeated\ Chebyshev}{\geq}\frac{4rs^2}{3.3^m}.\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^{m+1}.\frac{1}{3^{n-m-2}}.\left(\sum_{cyc}r_a\right)^{n-m-1}\overset{\substack{Nesbitt\\and\\Euler}}{\geq}$$

$$\frac{4rs^2}{3^{m+1}}\cdot\frac{3^{m+1}}{2^{m+1}}\cdot\frac{1}{3^{n-m-2}}\cdot(9r)^{n-m-1}\overset{\text{Mitrinovic}}{\geq}\frac{3^3.r^3.r^{n-m-1}}{2^{m-1}}\cdot\frac{1}{3^{n-m-2}}\cdot 3^{2n-2m-2}$$

$$=\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}\therefore\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}\therefore \text{combining } all \text{ cases, in } any\ \Delta\ ABC \text{ and } \forall\ m,n\in\mathbb{N}$$

$$\text{such that}: n\geq m-2,\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^n(r_b^2+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m}$$

$$\geq\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}},''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

1426.

In any Δ ABC, the following relationships hold :

$$\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{b}{a}}\right)\geq\sum_{cyc}\sqrt{\frac{a}{b}};\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right)\geq\sum_{cyc}\sqrt{\frac{w_a}{w_b}};\frac{R}{2r}\sum_{cyc}(g_a+h_a)>\sum_{cyc}n_a$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Tapas Das-India

$$n_a^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}$$

$$=s^2-\frac{4s(s-b)(s-c)}{a}=s^2-\frac{4s\cdot sr^2}{a(s-a)}=s^2-2r_ah_a\quad(1)$$

$$\text{also } 2r_a(n_a+h_a)\leq r_a^2+n_a^2+2r_ah_a=r_a^2+s^2\quad(\text{using (1)})$$

(AM-GM)

$$=s^2\left(\tan^2\frac{A}{2}+1\right)=\frac{s^2}{\cos^2\frac{A}{2}}=\frac{s\cdot bc}{s-a}=\frac{r_a}{r}\cdot 2Rh_a$$

$$n_a+h_a\leq R\cdot\frac{h_a}{r}\text{ or }\frac{n_a}{h_a}\leq\frac{R}{r}-1$$

$$\therefore n_a\leq\left(\frac{R}{r}-1\right)h_a\quad(\text{analog})$$

$$\therefore\sum n_a\leq\left(\frac{R}{r}-1\right)(h_a+h_b+h_c)$$

$$\frac{R}{2r}\cdot\sum(9a+h_a)\geq\frac{R}{2r}\sum(h_a+h_a)\quad(\because r_a\geq h_a)=\frac{R}{r}\sum h_a$$

We need to show

$$\frac{R}{r}\sum h_a\geq\left(\frac{R}{r}-1\right)(h_a+h_b+h_c)\text{ or }\frac{R}{r}\left(\sum h_a\right)-\left(\frac{R}{r}-1\right)\sum h_a\geq 0$$

$$\text{or }\sum h_a\geq 0\quad(\text{True})$$

1) $\sqrt{\frac{a}{b}}\overset{CBS}{\leq}\sqrt{\left(\sum a\right)\left(\sum\frac{1}{a}\right)}\leq\sqrt{2s\cdot\frac{9R}{4F}}=\sqrt{2s\cdot\frac{9R}{4rs}}=\sqrt{\frac{9R}{2r}}=\sqrt{\frac{9R^2}{2r\cdot R}}\overset{Euler}{\leq}\sqrt{\frac{9R^2}{9r^2}}=\frac{3R}{2r}$

Now

$$\frac{R}{2r}\sum\sqrt{\frac{b}{a}}\geq\frac{3R}{2r}$$

(AM-GM)

Note $\sqrt{\frac{b}{a}}\geq 3$ **(AM-GM)**

$$\frac{R}{2r}\sum\sqrt{\frac{a}{b}}\geq\sum\sqrt{\frac{a}{b}}$$

Note: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=\frac{s^2+r^2+4Rr}{4RF}\overset{Gerretsen}{\leq}\frac{4R^2+8Rr+4r^2}{4RF}=\frac{(R+r)^2}{RF}\overset{Euler}{\leq}\frac{9R^2}{4RF}=\frac{9R}{4F}$

$$\sum\sqrt{\frac{w_a}{w_b}}\leq\sqrt{3\left(\sum\frac{w_a}{w_b}\right)}\leq\sqrt{\frac{3R}{2r}\cdot 3}=\sqrt{\frac{9R}{2r}}=\sqrt{\frac{9R^2}{2R\cdot r}}\overset{Euler}{\leq}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

Note:

$$\sum\frac{w_a}{w_b}\leq\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{w_b^2}\right)}\leq\sqrt{\sum\left(s(s-a)\right)\sum\frac{1}{h_a^2}}\leq\sqrt{s^2\cdot\frac{\sum a^2}{4r^2s^2}}$$

$$\overset{Leibnitz}{\leq}\sqrt{\frac{s^2\cdot 9R^2}{9r^2s^2}}=\frac{3R}{2r}$$

We need to show

$$\frac{R^2}{4r^2}\sum\left(\sqrt{\frac{m_a}{m_b}}\right)\geq\frac{3R}{2r}\text{ or }\frac{R^2}{4r^2}\times 3\geq\frac{3R}{2r}$$

[Note: $\sqrt{\frac{m_a}{m_b}}\geq 3$**] (AM-GM) or** $\frac{R^2}{4r^2}\geq\frac{R}{2r}$

$R\geq 2r$ **(True) Euler**

$$\therefore\frac{R^2}{4r^2}\left(\sum\sqrt{\frac{m_a}{m_b}}\right)\geq\frac{3R}{2r}\geq\sum\sqrt{\frac{w_a}{w_b}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{a}{b}} \overset{A-G}{\leq} \sum_{cyc}\frac{\frac{a}{b}+1}{2} = \sum_{cyc}\frac{a+b+c}{2b} - \frac{1}{2}\sum_{cyc}\frac{c}{b} \overset{A-G}{\leq} \frac{2s}{2}\cdot\frac{s^2+4Rr+r^2}{4Rrs} - \frac{3}{2}$$
$$\overset{?}{\leq} \frac{3R}{2r} \Leftrightarrow \frac{3R+3r}{2r} \overset{?}{\geq} \frac{s^2+4Rr+r^2}{4Rr} \Leftrightarrow s^2+4Rr+r^2 \overset{?}{\leq} 6R^2+6Rr$$
$$\Leftrightarrow (s^2-4R^2-4Rr-3r^2) - 2(R+r)(R-2r) \overset{?}{\leq} 0 \to \text{true via Gerretsen and}$$
$$\text{Euler} \therefore \sum_{cyc}\sqrt{\frac{a}{b}} \leq \frac{3R}{2r} \overset{A-G}{\leq} \frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{b}{a}}\right) \therefore \boxed{\frac{R}{2r}\left(\sum_{cyc}\sqrt{\frac{b}{a}}\right) \geq \sum_{cyc}\sqrt{\frac{a}{b}}}$$
$$\sum_{cyc}\sqrt{\frac{w_a}{w_b}} \overset{A-G}{\leq} \sum_{cyc}\frac{\frac{w_a}{w_b}+1}{2} = \sum_{cyc}\frac{w_a+w_b+w_c}{2w_b} - \frac{1}{2}\sum_{cyc}\frac{w_c}{w_b} \overset{A-G}{\leq} \frac{\sum_{cyc} w_a}{2}\cdot\sum_{cyc}\frac{1}{w_a} - \frac{3}{2}$$
$$\leq \frac{\sum_{cyc} m_a}{2}\cdot\sum_{cyc}\frac{1}{h_a} - \frac{3}{2} \overset{Bager}{\leq} \frac{4R+r}{2r} - \frac{3}{2} \overset{?}{\leq} \frac{3R^2}{4r^2} \Leftrightarrow \frac{3R^2+6r^2}{4r^2} \overset{?}{\geq} \frac{4R+r}{2r}$$
$$\Leftrightarrow 3R^2-8Rr+4r^2 \overset{?}{\geq} 0 \Leftrightarrow (3R-2r)(R-2r) \overset{?}{\geq} 0 \to \text{true via Euler}$$
$$\therefore \sum_{cyc}\sqrt{\frac{w_a}{w_b}} \leq \frac{3R^2}{4r^2} \overset{A-G}{\leq} \frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right) \therefore \boxed{\frac{R^2}{4r^2}\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right) \geq \sum_{cyc}\sqrt{\frac{w_a}{w_b}}}$$
$$\text{Now, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) \overset{(m)}{=} an_a^2 + a(s-b)(s-c)$$
$$\text{and } b^2(s-b)+c^2(s-c) \overset{(n)}{=} ag_a^2 + a(s-b)(s-c) \text{ and } (m)+(n) \Rightarrow$$
$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$
$$= 2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2)$$
$$= 2(n_a^2+g_a^2) + a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2$$
$$= 2(n_a^2+g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow (b-c)^2 + 4s(s-a) + (b-c)^2$$
$$= 2(n_a^2+g_a^2) \Rightarrow n_a^2+g_a^2 \overset{(1)}{=} (b-c)^2 + 2s(s-a)$$
$$\text{Now, } \sum_{cyc}\frac{n_a}{R} = \sum_{cyc}\frac{2n_a h_a}{bc} \leq \sum_{cyc}\frac{2n_a g_a}{bc} \overset{A-G}{\leq} \sum_{cyc}\frac{n_a^2+g_a^2}{bc} \overset{via\ (1)}{=} \frac{(b-c)^2+2s(s-a)}{bc}$$
$$= \frac{1}{4Rrs}\left(\sum_{cyc}\left(a(b^2+c^2-2bc)\right) + 2s\sum_{cyc}\left(a(s-a)\right)\right)$$
$$= \frac{1}{4Rrs}\left(2s(s^2+4Rr+r^2) - 36Rrs + 2s\left(2s^2 - 2(s^2-4Rr-r^2)\right)\right)$$
$$= \frac{s^2-6Rr+3r^2}{2Rr} \Rightarrow 2r\sum_{cyc} n_a \leq s^2-6Rr+3r^2 \overset{?}{<} 2R.\sum_{cyc} h_a = s^2+4Rr+r^2$$
$$\Leftrightarrow 9Rr + r(R-2r) \overset{?}{>} 0 \therefore 2r\sum_{cyc} n_a < 2R.\sum_{cyc} h_a \leq R\sum_{cyc}(g_a+h_a)$$

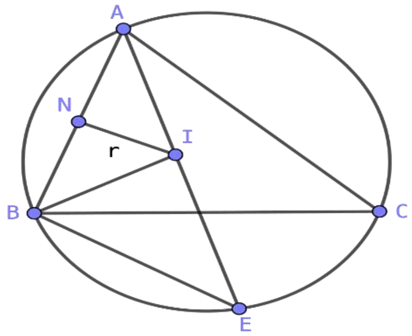
$$\therefore \boxed{\frac{R}{2r}\sum_{cyc}(g_a + h_a) > \sum_{cyc} n_a} \text{ (QED)}$$

1427. **Let $\Delta A'B'C'$ be the circumcevian triangle of incenter in acute ΔABC with r' - inradii. Prove that:**

$$r' \geq \frac{s}{3\sqrt{3}}, [A'B'C'] = \frac{1}{4}[I_aI_bI_c]$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Tapas Das-India

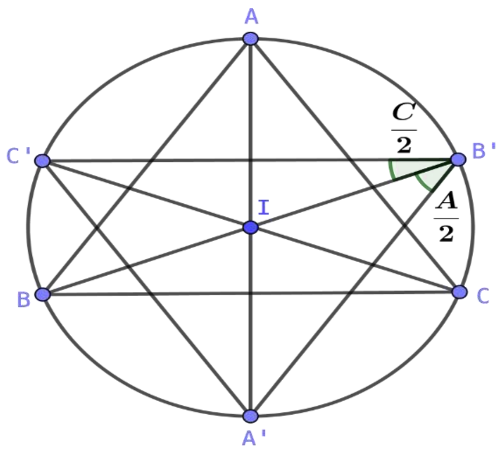


$AI = r\csc\frac{A}{2}$. From ΔANI: $IN =$ perpendicular from I on $AB = r$ (in-radius)

Again $\angle BIE = \angle BAI + \angle ABI = \angle CAE + \angle IBC = \angle CBE + \angle IBC = \angle EBI$

$$\therefore IE = EB = 2R\sin\frac{A}{2}$$

$$\therefore IE = 2R\sin\frac{A}{2} \quad \text{(analog)}$$



From $\Delta IA'B'$

$$\angle AA'B' = \angle ABB' = \frac{B}{2}, \qquad \angle BB'A' = \angle BAA' = \frac{A}{2}$$

$$\angle A'IB' = \pi - \left(\frac{A+B}{2}\right)$$

From $\Delta A'IB'$

$$\frac{A'B'}{\sin \angle A'IB'} = \frac{A'I}{\sin\frac{A}{2}} \Rightarrow \frac{A'B'}{\sin\left(\pi - \frac{A+B}{2}\right)} = \frac{2R\sin\frac{A}{2}}{\sin\frac{A}{2}} = 2R$$

$$A'B' = 2R\sin\frac{A+B}{2} = 2R\cos\frac{C}{2}$$

(analog)

$$[A'B'C'] = \frac{1}{2}B'C'\cdot A'B'\cdot \sin\angle A'B'C' = \frac{1}{2}2R\cos\frac{C}{2}\cdot 2R\cos\frac{A}{2}\cdot\sin\frac{A+C}{2}$$

$$[A'B'C'] = 2R^2\cos\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2} \quad (1)$$

Now $[A'B'C'] = r's'$ (s' = semi – perimeter of $\Delta A'B'C'$)

or $2R^2\prod\cos\frac{A}{2} = \frac{r'}{2}2R\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$

$$r' = \frac{2R\prod\cos\frac{A}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}$$

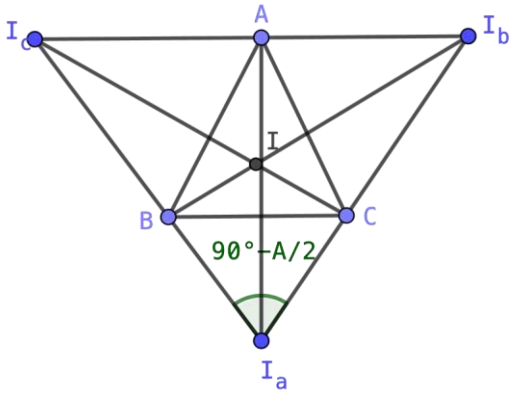
$$\therefore r' \geq \frac{2R\cdot\frac{s}{4R}}{\frac{3\sqrt{3}}{2}} = \frac{s}{3\sqrt{3}}$$

Note:

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \leq 3\cos\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right) \overset{Jensen}{=} 3\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$$

$f(x) = \cos x \ \therefore f'(x) = -\sin x, f''(x) = -\cos x < 0, \therefore f$ **is concave**

2nd part:



Since ΔABC is orthic triangle of $\Delta I_aI_bI_c$ then

$$BC = a = I_aI_c\cdot\cos\left(90^\circ - \frac{A}{2}\right) = I_bI_c\sin\frac{A}{2},\qquad a = 2R\sin A = I_bI_c\sin\frac{A}{2}$$

$$2R\,2\cdot\sin\frac{A}{2}\cos\frac{A}{2} = I_bI_c\sin\frac{A}{2},\ \ I_bI_c = 4R\cos\frac{A}{2}\ \ \text{(analog)}$$

$$[I_aI_bI_c] = \frac{1}{2}\cdot 4R\cos\frac{B}{2}4R\cos\frac{C}{2}\sin\left(90^\circ - \frac{A}{2}\right) = 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$[A'B'C'] = 2R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\text{(From (1))} \therefore [A'B'C'] = \frac{1}{4}[I_aI_bI_c]$$

1428. In ΔABC the following relationship holds:

$$\sum\frac{w_a}{w_b} + \frac{R^3}{r^3} \geq 8 + \sum\frac{w_b}{w_a}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\sum\frac{w_b}{w_a} \leq \sum\frac{\sqrt{s(s-b)}}{w_a} \leq \sum\frac{\sqrt{s(s-b)}}{h_a}$$

$$\overset{CBS}{\leq}\sqrt{\left(\sum s(s-b)\right)\left(\sum\frac{1}{h_a^2}\right)} = \sqrt{s^2\cdot\frac{\sum a^2}{4r^2s^2}} \overset{Leibnitz's}{\leq} \sqrt{\frac{9R^2}{4r^2}} = \frac{3R}{2r}$$

$$\therefore 8 + \sum\frac{w_b}{w_a} \leq \frac{3R}{2r} + 8$$

Note: $\sum\frac{w_b}{w_a} \geq 3$ (AM-GM)

We need to show

$$3+\frac{R^3}{r^3}\geq\frac{3R}{2r}+8 \text{ or } \frac{R^3}{r^3}\geq\frac{3R}{2r}+5$$

$$\text{or } x^3\geq\frac{3}{2}x+5 \quad \left(\frac{R}{r}=x\geq 2\right)$$

$$\text{or } 2x^3-3x-10\geq 0$$

$$2x^3-4x^2+4x^2-8x+5x-10\geq 0$$

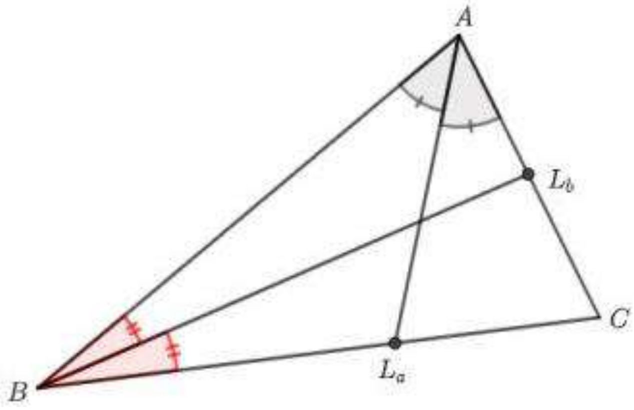
$$\text{or } 2x^2(x-2)+4x(x-2)+5(x-2)\geq 0$$

$$(x-2)(2x^2+4x+5)\geq 0 \quad \text{(True)}$$

as $x\geq 2.$

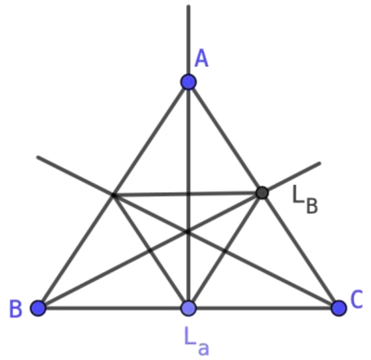
1429. Prove that:

$$L_aL_b=\frac{4S\sqrt{R(R+2r_c)}}{(a+c)(b+c)}$$



Proposed by Aissa Hiyab-Morocco

Solution by Tapas Das-India



$$\frac{BL_a}{CL_a}=\frac{c}{b}$$

$$\therefore CL_a=\frac{ab}{b+c}$$

$$CL_B=\frac{ab}{c+a}$$

$$\therefore L_aL_b^2=\frac{a^2b^2}{(b+c)^2}+\frac{a^2b^2}{(c+a)^2}-2\cos C\ CL_a\cdot CL_b$$

$$=\frac{a^2b^2}{(b+c)^2}+\frac{a^2b^2}{(c+a)^2}-2\frac{a^2b^2}{(b+c)(c+a)}\cdot\frac{a^2+b^2-c^2}{2ab}$$

$$=\frac{a^2b^2(c+a)^2+a^2b^2(b+c)^2-ab(b+c)(c+a)(a^2+b^2-c^2)}{(b+c)^2(c+a)^2}$$

$$=\frac{a^2b^2(c^2+a^2+2ca+b^2+c^2+2bc)-ab(bc+ab+c^2+ac)(a^2+b^2-c^2)}{(b+c)^2(c+a)^2}$$

$$=\frac{a^2b^2(c^2+a^2+2ca+b^2+c^2+2bc)-a^2b^2(a^2+b^2-c^2)-ab(c^2+ac+bc)(a^2+b^2-c^2)}{(b+c)^2(c+a)^2}\quad (1)$$

Now, $a^2b^2(c^2+a^2+b^2+c^2+2ca+2bc)-a^2b^2(a^2+b^2-c^2)-$

$$-ab(c^2+ac+bc)(a^2+b^2-c^2)$$

$$=a^2b^2[c^2+2(bc+ca+c^2)]-(bc+ca+c^2)(a^2+b^2-c^2)ab$$

$$=a^2b^2c^2+(bc+ca+c^2)(2a^2b^2-(a^2+b^2-c^2)ab)$$

$$=a^2b^2c^2+2sc\cdot ab\,[2ab-(a^2+b^2-c^2)]$$

$$=a^2b^2c^2+2sc\cdot ab[c^2-(a-b)^2]$$

$$=a^2b^2c^2+2sc\cdot ab(c+a-b)(c-a+b)$$

$$=16F^2R^2+2s(4RF)(2s-2b)(2s-2a)=16F^2R^2+32sRF(s-b)(s-a)$$

$$=16F^2R^2+\frac{32F^2\cdot FR}{(s-c)}=16F^2R^2+32r_c\cdot F^2R$$

$$=16F^2R^2+32r_cF^2R=16F^2R(R+2r_c)$$

$\therefore$ From (1) we get,

$$L_aL_b^2=\frac{16F^2R(R+2r_c)}{(b+c)^2(c+a)^2},\qquad L_aL_b=\frac{4F\sqrt{R(R+2r_c)}}{(b+c)(c+a)}$$

1430. **In ΔABC the following relationship holds:**

$$\sum \frac{m_a}{m_b m_c (m_b + m_c)} \ge \frac{16r^3}{3R^5}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\sum \frac{m_a}{m_b m_c (m_b + m_c)} = \frac{1}{m_a m_b m_c} \sum \frac{m_a^2}{m_b + m_c}$$

$$\ge \frac{1}{m_a m_b m_c} \frac{(m_a + m_b + m_c)^2}{2(m_a + m_b + m_c)} = \frac{1}{2m_a m_b m_c}(m_a + m_b + m_c)$$

Note:

$$m_a m_b m_c \le \frac{R^3 h_a h_b h_c}{8r^3} = \frac{R^3}{8r^3}\frac{8F^3}{abc} = \frac{R^3 F^3}{r^3 \cdot 4RF} = \frac{R^3 F^2}{4Rr^3}$$

Note: $m_a + m_b + m_c \ge 9r$

$$\therefore \sum \frac{m_a}{m_b m_c (m_b + m_c)} \ge \frac{1}{2m_a m_b m_c}(m_a + m_b + m_c) \ge \frac{1}{2}\frac{4Rr^3}{R^3 F^2} \cdot 9r$$

$$\overset{Euler}{\ge} \frac{1}{2}\frac{4(2r)r^3 \cdot 9r}{R^3 r^2 s^2} \ge \frac{1}{2} \cdot \frac{4(2r)r^3 \cdot 9r}{R^3 r^2 \cdot \frac{27}{4} R^2}$$

$$\left(s^2 \le \frac{27}{4} R^2\right)$$

$$= \frac{1}{2} \cdot 8r \cdot 9r^4 \times \frac{4}{27R^5 r^2} = \frac{16r^3}{3R^5}$$

Note:

$$m_a + m_b + m_c \ge \sqrt{s(s-a)} + \sqrt{s(s-b)} + \sqrt{s(s-c)}$$

$$\overset{AM-GM}{\ge} 3\left(\sqrt{s^3(s-a)(s-b)(s-c)}\right)^{\frac{1}{3}} = 3\left(\sqrt{s^3 \cdot sr^2}\right)^{\frac{1}{3}} = 3(s^2 r)^{\frac{1}{3}}$$

$$(s^2 \ge 27r^2) \ge 3(27r^2 \cdot r)^{\frac{1}{3}} = 9r$$

1431. **Prove that in any triangle ABC with usual notations, $x, y \in \mathbb{R}_+^*$, $xy = 1$, holds the following inequalities:**

i) $(ax^2 + b + cy^2)(ay^2 + b + cx^2) \ge 4s^2$

ii) $(ax+b+cy)(ay+b+cx)\geq 4s^2$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Tapas Das – India

i) $(ax^2+b+cy^2)(ay^2+b+cx^2)$

$$\overset{Cauchy-Schwarz}{\geq}(axy+b+cxy)^2=(a+b+c)^2\ \ (\because xy=1)=(2s)^2=4s^2$$

ii) $(ax+b+cy)(ay+b+cx)\geq\left(a\sqrt{xy}+b+c\sqrt{xy}\right)^2$ (Cauchy – Schwarz)

$$=(a+b+c)^2=(2s)^2=4s^2$$

$$\because (xy=1)$$

1432. In ΔABC the following relationships holds:

$$\left(\frac{\left(\sqrt{m_a}\right)^3+\left(\sqrt{w_a}\right)^3}{2}\right)^2\geq\left(2\sqrt{s(s-a)}\right)^3$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

Note: $m_a\geq\frac{b+c}{2}\cos\frac{A}{2}$

$$w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$$

$$\therefore m_aw_a=\frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{2bc}{b+c}\cdot\cos\frac{A}{2}=bc\cos^2\frac{A}{2}=bc\frac{s(s-a)}{bc}=s(s-a)$$

$$\therefore\left[\frac{\left(\sqrt{m_a}\right)^3+\left(\sqrt{w_a}\right)^3}{2}\right]^2\overset{AM-GM}{\geq}\left[\sqrt{\left(\sqrt{m_a}\right)^3\cdot\left(\sqrt{w_a}\right)^3}\right]^2=\left(\sqrt{m_a\cdot w_a}\right)^3\geq\left(\sqrt{s(s-a)}\right)^3$$

1433. In ΔABC the following relationship holds:

$$4r^2(5r-r)\leq\sum r_a\cdot IA^2\leq 4r(R+r)^2$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

$$\sum r_aI_a^2=\sum r_ar^2\csc^2\frac{A}{2}=\sum r_ar^2\left(1+\cot^2\frac{A}{2}\right)$$

$$= \sum r_a r^2\left(1+\frac{s^2}{r_a^2}\right) = \sum (r_a r^2) + \left(\frac{r^2 s^2}{r_a}\right) = r^2\sum r_a + \sum \frac{r^2 s^2}{r\cdot s}(s-a)$$

$$= r^2\sum r_a + \sum rs(s-a) = r^2(4R+r) + (3s^2-2s^2)r = r^2(4R+r)+s^2 r$$

$$\overset{Gerretsen's}{\le} r^2(4R+r) + (4R^2+4Rr+3r^2)r$$

$$= r(4R^2+4Rr+3r^2+4Rr+r^2) = 4r(R+r)^2$$

Again, $\sum r_a \cdot IA^2 = r^2(4R+r)+s^2 r \overset{Gerretsen's}{\ge} r^2(4R+r)+r(16Rr-5r^2)$

$$= r[4Rr+r^2+16Rr-5r^2] = r[20Rr-4r^2] = 4r^2(5R-r)$$

1434. **In ΔABC the following relationship holds:**

$$\sum a^{2n} \ge 3\left(\frac{4F}{3}\sqrt{\sum \frac{a^2}{b^2}}\right)^n, n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

Oppenheim, $\forall x,y,z>0$ in ΔABC

$$a^2x+b^2y+c^2z \ge 4F\sqrt{xy+yz+zx}$$

Let $x=\frac{b^2}{a^2}, y=\frac{c^2}{b^2}, z=\frac{a^2}{c^2}$

$$\therefore a^2+b^2+c^2 = a^2\cdot\frac{b^2}{a^2}+b^2\cdot\frac{c^2}{b^2}+c^2\cdot\frac{a^2}{c^2} \ge 4F\sqrt{\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}+\frac{c^2}{b^2}\cdot\frac{a^2}{c^2}+\frac{a^2}{c^2}\cdot\frac{b^2}{a^2}}$$

$$= 4F\sqrt{\frac{c^2}{a^2}+\frac{a^2}{b^2}+\frac{b^2}{c2}} \quad (1)$$

$$\therefore a^{2n}+b^{2n}+c^{2n} = (a^2)^n+(b^2)^n+(c^2)^n \overset{CBS}{\ge} \frac{3(a^2+b^2+c^2)^n}{3^n}$$

$$\ge 3\left[\frac{4F}{3}\sqrt{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}\right]^n \text{(Using (1))}$$

1435. **In acute ΔABC the following relationship holds:**

$$\frac{2R^2}{r^2}+1 \le \sum\frac{(1+\sec A)^2}{\tan^2 A} \le \left(\frac{2R}{r}-1\right)^2$$

Proposed by Marin Chirciu – Romania

Solution by Tapas Das – India

1st part:

$$\sum\frac{(1+\sec A)^2}{\tan^2 A}=\sum\frac{(1+\sec A)^2}{(\sec^2 A-1)}=\sum\frac{1+\sec A}{\sec A-1}=\sum\frac{1+\cos A}{1-\cos A}=\sum\cot\frac{A}{2}$$

$$\therefore\sum\cot^2\frac{A}{2}=\sum\frac{1}{\tan^2\frac{A}{2}}=\frac{\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\prod\tan^2\frac{A}{2}}=\frac{1-\frac{2r^2+8Rr}{s^2}}{\frac{r^2}{s^2}}$$

$$=\frac{s^2-2r^2-8Rr}{r^2}\overset{Gerretsen's}{\leq}\frac{4R^2-4Rr+r^2}{r^2}=\left(\frac{2R}{r}-1\right)^2$$

2nd part:

$$\sum\frac{(1+\sec A)^2}{\tan^2 A}=\sum\cot^2\frac{A}{2}=\sum\frac{1}{\tan^2\frac{A}{2}}=\frac{\sum\tan^2\frac{A}{2}\cdot\tan^2\frac{B}{2}}{\prod\tan^2\frac{A}{2}}=\frac{s^2-2r^2-8Rr}{r^2}$$

$$\overset{\substack{(Walker's)\\ since\ triangle\ is\ acute}}{\geq}\frac{2R^2+8Rr+3r^2-2r^2-8Rr}{r^2}=\frac{2R^2}{r^2}+1$$

1436. In any acute triangle ABC holds :

$$m_a\sqrt{\cot A}+m_b\sqrt{\cot B}+m_c\sqrt{\cot C}>6r,$$

where m_a,m_b,m_c are the medians and r is the inradius of the triangle.

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we may assume that $a\leq b\leq c$.
We have $m_a\geq m_b\geq m_c$ and $\sqrt{\cot A}\geq\sqrt{\cot B}\geq\sqrt{\cot C}$,
then by Chebyshev's inequality, we get

$$m_a\sqrt{\cot A}+m_b\sqrt{\cot B}+m_c\sqrt{\cot C}\geq\frac{1}{3}(m_a+m_b+m_c)\left(\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\right).$$

Since $m_a\geq h_a$ (and analogs) and
$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$, then by using AM − HM inequality, we have

$$m_a+m_b+m_c\geq h_a+h_b+h_c\geq\frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=9r.$$

Now, let $x:=\cot A$, $y:=\cot B$, $z:=\cot C$.
We have $x\geq y\geq z>0$, $xy+yz+zx=1$ and by using
AM − GM inequality, we obtain

$$\left(\frac{\sqrt{x}+\left(\sqrt{y}+\sqrt{z}\right)}{2}\right)^4\geq\left(\sqrt{x}\left(\sqrt{y}+\sqrt{z}\right)\right)^2=xy+2x\sqrt{yz}+zx>xy+yz+zx=1,$$

then $\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}=\sqrt{x}+\sqrt{y}+\sqrt{z}>2.$

Therefore

$$m_a\sqrt{\cot A}+m_b\sqrt{\cot B}+m_c\sqrt{\cot C}>\frac{1}{3}.9r.2=6r.$$

1437. O **– the circumcenter of** ΔABC **lies on the incircle of** ΔABC**. Prove that:**

$$8\sqrt{2}+\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}>12$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

$$\cos\frac{B-C}{2}=\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\qquad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

Using Ravi's transformation

$$a=y+z,b=z+x,c=x+y$$

Now we need to show $\cos\frac{B-C}{2}\geq\sqrt{\frac{2r}{R}}$

Now the inequality is equivalent to with $s=x+y+z$

$$\cos\frac{B-C}{2}=\sqrt{\frac{(s-b)(s-c)}{ac\cdot ab}}+\sqrt{\frac{(s-a)(s-c)}{ac}\cdot\frac{(s-b)(s-a)}{ab}}$$

This is equivalent to: $(2x+y+z)^2\geq 8x(y+z)$

This is true using AM-GM

$$\therefore\cos\frac{B-C}{2}\geq\sqrt{\frac{2r}{R}}$$

$$\therefore\cos\frac{B-C}{2}\cdot\cos\frac{A-B}{2}\cdot\cos\frac{C-A}{2}\geq\frac{2r}{R}\sqrt{\frac{2r}{R}}$$

Now incircle passes through circumcentre

$$\therefore OI=r\Rightarrow\sqrt{R^2-2Rr}=r$$

$$\therefore \left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right) - 1 = 0$$

$$\therefore \frac{r}{R} = \frac{-2 \pm \sqrt{4+4}}{2} \therefore \frac{r}{R} = (\sqrt{2}-1) \quad \left(\frac{r}{R} > 0\right)$$

We need to show

$$8\sqrt{2} + \prod \cos\frac{A-B}{2} > 12 \text{ or } 8\sqrt{2} + \frac{2r}{R}\sqrt{\frac{2r}{R}} > 12$$

$$\text{or } 8\sqrt{2} + 2(\sqrt{2}-1)\sqrt{2(\sqrt{2}-1)} > 12 \text{ or } 4\sqrt{2} + (\sqrt{2}-1)\sqrt{2(\sqrt{2}-1)} > 6$$

$$\text{or } 4 + (\sqrt{2}-1)\sqrt{\sqrt{2}-1} > 3\sqrt{2} \text{ or } (4-3\sqrt{2}) > -(\sqrt{2}-1)\sqrt{\sqrt{2}-1}$$

$$\text{or } 16 + 18 - 24\sqrt{2} > (\sqrt{2}-1)^2(\sqrt{2}-1) \text{ or } 34 - 24\sqrt{2} > 3\sqrt{2} - 7 + 2\sqrt{2}$$

$$\text{or } 41 > 29\sqrt{2} \quad \text{True.}$$

$$\prod \cos\frac{A-B}{2} = \frac{(a+b)(b+c)(c+a)}{abc}\prod \sin\frac{A}{2}$$

$$= \frac{(a+b+c)(ab+bc+ca) - abc}{abc} \cdot \frac{r}{4R} = \frac{s^2 + r^2 + 2Rr}{8R^2}$$

When incircle passes through circumcircle

$$OI = r \Rightarrow \sqrt{R^2 - 2Rr} = r \text{ or } \left(\frac{r}{R}\right)^2 + 2\left(\frac{r}{R}\right) - 1 = 0$$

$$\text{or } \frac{r}{R} = \frac{-2 \pm \sqrt{4+4}}{2} \therefore \frac{r}{R} = \sqrt{2} - 1$$

$$\left(\frac{r}{R} > 0\right)$$

We need to show:

$$8\sqrt{2} + \frac{s^2 + r^2 + 2Rr}{8R^2} > 12$$

$$\text{Or } 8\sqrt{2} + \frac{27r^2 + r^2 + 2Rr}{8R^2} > 12 \quad \textbf{(Mitrinovic)}$$

$$\text{or } 8\sqrt{2} + \frac{27(\sqrt{2}-1)^2 \cdot R^2 + (\sqrt{2}-1)^2 R^2 + 2R^2(\sqrt{2}-1)}{8R^2} > 12$$

$$\left(\text{using } \frac{R}{r} = \sqrt{2} - 1\right)$$

$$\text{or } 64\sqrt{2} + 28(\sqrt{2}-1)^2 + 2(\sqrt{2}-1) > 96 \text{ or } 64\sqrt{2} + 28(3 - 2\sqrt{2}) + 2\sqrt{2} - 2 > 96$$

or $64\sqrt{2}+84-56\sqrt{2}+2\sqrt{2}-2>96$ or $10\sqrt{2}+82>96$ or $10\sqrt{2}>14$

or $\left(10\sqrt{2}\right)^2>(14)^2$ or $250>196$ (True)

$$\cos\frac{B-C}{2}=\frac{h_a}{w_a}\text{ (analog)}$$

We show that

$\prod\cos\frac{B-C}{2}\geq 8\prod\sin\frac{A}{2}$ or $\frac{h_ah_bh_c}{w_aw_bw_c}\geq 8\cdot\frac{r}{4R}$ or $Rh_ah_bh_c\geq 2r(w_aw_bw_c)$

or $R\cdot\frac{(2F)^3}{abc}\geq 2r\sqrt{s(s-a)}\sqrt{s(s-b)}\sqrt{s(s-c)}$ or $8F^3R\geq 2r^2s^2\cdot 4Rrs$

or $8r^3s^3R\geq 8r^3s^3R$ (True)

$$\therefore\prod\cos\frac{B-C}{2}\geq 8\prod\sin\frac{A}{2}=2\cdot\frac{r}{R}$$

$$=2\left(\sqrt{2}-1\right)$$

Now, incircle passes through circumcircle

$$\therefore OI=r\Rightarrow\sqrt{R^2-2Rr}=r\Rightarrow\left(\frac{r}{R}\right)^2+2\left(\frac{r}{R}\right)-1=0$$

$$\therefore\frac{r}{R}=\frac{-2\pm\sqrt{4+4}}{2}\ \therefore\frac{r}{R}=\sqrt{2}-1$$

$$\left(\frac{r}{R}>0\right)$$

We need to show

$8\sqrt{2}+\prod\cos\frac{B-C}{2}>12$ or $8\sqrt{2}+2\left(\sqrt{2}-1\right)>12$

or $10\sqrt{2}>14$ or $250>196$ (True)

1438. In any $\Delta\,ABC$ and $\forall\, n\in\mathbb{N}$, the following relationship holds :

$$\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)}\geq\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{m_a}\overset{\text{Bergstrom}}{\geq}\frac{9}{\sum_{cyc}m_a}\overset{\text{Leuenberger}}{\geq}\frac{9}{4R+r}\overset{\text{Euler}}{\geq}\frac{9}{\frac{9R}{2}}\Rightarrow\sum_{cyc}\frac{1}{m_a}\overset{(i)}{\geq}\frac{2}{R}$$

Case 1 $n = 1$ and then : $\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)}$

$$=\sum_{cyc}\frac{m_a}{m_b+m_c}+\sum_{cyc}\frac{m_bm_c}{m_am_b+m_am_c}\overset{Nesbitt}{\geq}\frac{3}{2}+\frac{3}{2}=3=\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1}$$

Case 2 $n\in\mathbb{N}-\{1\}$ and then : $\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)}$

$$=\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^{n-2}}{m_b+m_c}+\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^{n}}{\frac{m_b+m_c}{m_bm_c}}\overset{Holder}{\geq}\frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^{n-2}}{3^{n-4}*2*\sum_{cyc}m_a}+\frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^{n}}{3^{n-2}*2*\sum_{cyc}\frac{1}{m_a}}$$

$$\overset{\substack{\text{via (i)}\\ and\\ \text{Leuenberger + Euler}}}{\geq}\frac{\left(\frac{2}{R}\right)^{n-2}}{3^{n-4}*2*\frac{9R}{2}}+\frac{\left(\frac{2}{R}\right)^{n-1}}{3^{n-2}*2}=\frac{2^{n-2}}{3^{n-2}*R^{n-1}}+\frac{2^{n-2}}{3^{n-2}*R^{n-1}}$$

$=\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1}$ and combining both cases, in any Δ ABC and $\forall\, n\in\mathbb{N}$,

$$\frac{m_a^2+m_bm_c}{m_a^n(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^n(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^n(m_a+m_b)}$$

$$\geq\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1}, ''='' \text{ iff } \Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we may assume that $m_a\geq m_b\geq m_c$.

Since

$$\frac{1}{m_a{}^n(m_b+m_c)}\leq\frac{1}{m_b{}^n(m_c+m_a)}\leq\frac{1}{m_c{}^n(m_a+m_b)},$$

then by the Generalized Schur inequality, we have

$$\sum_{cyc}\frac{m_a{}^2+m_bm_c}{m_a{}^n(m_b+m_c)}=\sum_{cyc}\left(\frac{(m_a-m_b)(m_a-m_c)}{m_a{}^n(m_b+m_c)}+\frac{1}{m_a{}^{n-1}}\right)\geq\sum_{cyc}\frac{1}{m_a{}^{n-1}}$$

$$\overset{H\ddot{o}lder}{\geq}\frac{3^n}{(m_a+m_b+m_c)^{n-1}}\overset{Gotman}{\geq}\frac{3^n}{\left(\frac{9R}{2}\right)^{n-1}}=\frac{1}{3^{n-2}}\cdot\left(\frac{2}{R}\right)^{n-1},$$

as desired. Equality holds iff ΔABC is equilateral.

1439. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2+m_bm_c}{m_a^5(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^5(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^5(m_a+m_b)}\geq\frac{16}{27R^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2+m_bm_c}{m_a^5(m_b+m_c)}+\frac{m_b^2+m_cm_a}{m_b^5(m_c+m_a)}+\frac{m_c^2+m_am_b}{m_c^5(m_a+m_b)}$$

$$=\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^3}{m_b+m_c}+\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^5}{\frac{m_b+m_c}{m_bm_c}}\overset{Holder}{\geq}\frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^3}{6\sum_{cyc}m_a}+\frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^5}{27*2\sum_{cyc}\frac{1}{m_a}}$$

$$\overset{Bergstrom}{\geq}\frac{\left(\frac{9}{\sum_{cyc}m_a}\right)^3}{6\sum_{cyc}m_a}+\frac{\left(\frac{9}{\sum_{cyc}m_a}\right)^4}{54}\overset{Leuenberger}{\geq}\frac{729}{6(4R+r)^4}+\frac{729*9}{54(4R+r)^4}$$

$$\overset{Euler}{\geq}\frac{729*2^4}{6*729*9R^4}+\frac{729*9*2^4}{54*729*9R^4}=\frac{16}{27R^4},''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sum_{cyc}\frac{m_a{}^2+m_bm_c}{m_a{}^5(m_b+m_c)}\geq\sum_{cyc}\frac{2m_a\sqrt{m_bm_c}}{m_a{}^5(m_b+m_c)}$$

$$\geq 2.3\sqrt[3]{\frac{m_a{}^2m_b{}^2m_c{}^2}{m_a{}^5m_b{}^5m_c{}^5(m_a+m_b)(m_b+m_c)(m_c+m_a)}}$$

$$=\frac{6}{m_am_bm_c\sqrt[3]{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}$$

$$\geq\frac{6}{\left(\frac{m_a+m_b+m_c}{3}\right)^3.\frac{(m_a+m_b)+(m_b+m_c)+(m_c+m_a)}{3}}=\frac{243}{(m_a+m_b+m_c)^4}$$

$$\overset{Leuenberger}{\geq}\frac{243}{(4R+r)^4}\overset{Euler}{\geq}\frac{243}{\left(\frac{9R}{2}\right)^4}=\frac{16}{27R^4},$$

as desired. Equality holds iff ΔABC is equilateral.

1440. In any Δ ABC, the following relationship holds :

$$\frac{w_a(w_b^2+w_c^2)}{w_a^2+w_bw_c}+\frac{w_b(w_c^2+w_a^2)}{w_b^2+w_cw_a}+\frac{w_c(w_a^2+w_b^2)}{w_c^2+w_aw_b}\geq 9r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

Firstly, we shall prove that $\forall\, x,y,z>0,\ \prod_{cyc}(y^2+z^2)\geq\prod_{cyc}(x^2+yz)$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^2y^4\overset{(i)}{\geq}xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3$$

$$\text{LHS of (i)} = \sum_{cyc}\frac{x^4y^2+x^4z^2}{2}+\sum_{cyc}\frac{x^4y^2+x^2y^4}{2}\overset{A-G}{\geq}\sum_{cyc}x^4yz+\sum_{cyc}x^3y^3$$

$$= xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3 \Rightarrow \text{(i) is true} \therefore \forall\, x,y,z>0, \frac{\prod_{cyc}(y^2+z^2)}{\prod_{cyc}(x^2+yz)}\geq 1 \rightarrow (1)$$

$$\text{Now}, \frac{w_a(w_b^2+w_c^2)}{w_a^2+w_bw_c}+\frac{w_b(w_c^2+w_a^2)}{w_b^2+w_cw_a}+\frac{w_c(w_a^2+w_b^2)}{w_c^2+w_aw_b}\overset{A-G}{\geq}$$

$$3\sqrt[3]{w_aw_bw_c.\frac{\prod_{cyc}(w_b^2+w_c^2)}{\prod_{cyc}(w_a^2+w_bw_c)}}\overset{\text{via (1)}}{\geq} 3.\sqrt[3]{w_aw_bw_c}\geq 3.\sqrt[3]{h_ah_bh_c}=3.\sqrt[3]{\frac{2r^2s^2}{R}}$$

$$\overset{\text{Gerretsen}}{\geq} 3.\sqrt[3]{\frac{r^2.(27Rr+5r(R-2r))}{R}}\overset{\text{Euler}}{\geq} 3.\sqrt[3]{\frac{r^2.27Rr}{R}}=9r \therefore \text{in any } \Delta\,ABC,$$

$$\frac{w_a(w_b^2+w_c^2)}{w_a^2+w_bw_c}+\frac{w_b(w_c^2+w_a^2)}{w_b^2+w_cw_a}+\frac{w_c(w_a^2+w_b^2)}{w_c^2+w_aw_b}\geq 9r,$$

$'' ='' $ iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\sum_{cyc}\frac{w_a({w_b}^2+{w_c}^2)}{{w_a}^2+w_bw_c}\overset{CBS}{\geq}\sum_{cyc}\frac{w_a({w_b}^2+{w_c}^2)}{\sqrt{({w_a}^2+{w_b}^2)({w_a}^2+{w_c}^2)}}\overset{AM-GM}{\geq} 3\sqrt[3]{\prod_{cyc}\frac{w_a({w_b}^2+{w_c}^2)}{\sqrt{({w_a}^2+{w_b}^2)({w_a}^2+{w_c}^2)}}}$$

$$=3\sqrt[3]{w_aw_bw_c}\overset{w_a\geq h_a \text{ (and analogs)}}{\geq} 3\sqrt[3]{h_ah_bh_c}\overset{GM-HM}{\geq}\frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}=9r.$$

as desired. Equality holds iff ΔABC is equilateral.

1441. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{m_a}{m_b\left(5(m_a^2+m_b^2)+m_a(6m_b+11m_c)\right)}\geq\frac{4}{81R^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0, \sum_{cyc}\frac{x}{y(5(x^2+y^2)+x(6y+11z))}$$

$$=\sum_{cyc}\frac{x^2}{5x^3y+5xy^3+6x^2y^2+11x.xyz}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc}x\right)^2}{5\sum_{cyc}x^3y+5\sum_{cyc}xy^3+6\sum_{cyc}x^2y^2+11xyz\sum_{cyc}x}\overset{?}{\geq}\frac{1}{\left(\sum_{cyc}x\right)^2}$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^4\overset{?}{\geq}5\sum_{cyc}\left(xy\left(\sum_{cyc}x^2-z^2\right)\right)+6\sum_{cyc}x^2y^2+11xyz\sum_{cyc}x$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^4\underset{(*)}{\overset{?}{\geq}}5\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+6\sum_{cyc}x^2y^2+6xyz\sum_{cyc}x$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\to(1)\Rightarrow x=s-X, y=s-Y,$ $z=s-Z$ and such substitutions $\Rightarrow\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)$

$$\Rightarrow\sum_{cyc}xy=4Rr+r^2\to(2)\text{ and }\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via (1) and (2)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}x^2=s^2-8Rr-2r^2\to(3)\text{ and also, }\sum_{cyc}x^2y^2$$

$$=\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)\overset{\text{via (1) and (2)}}{=}(4Rr+r^2)^2-2\left(\prod_{cyc}(s-X)\right).s$$

$$=(4Rr+r^2)^2-2r^2s.s\Rightarrow\sum_{cyc}x^2y^2=r^2((4R+r)^2-2s^2)\to(4)$$

$$\therefore\text{ via }(1),(2),(3),(4),(*)\Leftrightarrow s^4\geq 5(4Rr+r^2)(s^2-8Rr-2r^2)$$

$$+6r^2((4R+r)^2-2s^2)+6\left(\prod_{cyc}(s-X)\right)s$$

$$=5(4Rr+r^2)(s^2-8Rr-2r^2)+6r^2((4R+r)^2-2s^2)+6r^2s$$

$$\Leftrightarrow s^4-(20Rr-r^2)s^2+4r^2(4R+r)^2\overset{(**)}{\geq}0\text{ and }\because(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)\geq(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(4R-3r)s^2\overset{(***)}{\geq}r(64R^2-64Rr+7r^2)$$

Now, $(4R-3r)s^2\overset{\text{Rouche}}{\geq}(4R-3r)\left(2R^2+10Rr-r^2-2(R-2r)*\sqrt{R^2-2Rr}\right)$

$$\overset{?}{\geq}r(64R^2-64Rr+7r^2)\Leftrightarrow(4R-3r)(2R^2+10Rr-r^2)-r(64R^2-64Rr+7r^2)$$

$$\overset{?}{\geq}2(4R-3r)(R-2r)*\sqrt{R^2-2Rr}$$

$\Leftrightarrow 2(R-2r)(4R^2-7Rr+r^2)\overset{?}{\geq}2(4R-3r)(R-2r)*\sqrt{R^2-2Rr}$ and

proving it will be done if we can prove : $4R^2-7Rr+r^2>(4R-3r)*\sqrt{R^2-2Rr}$

$$\left(\because R-2r\overset{\text{Euler}}{\geq}0\right)\Leftrightarrow(4R^2-7Rr+r^2)^2>(R^2-2Rr)(4R-3r)^2$$

$$\Leftrightarrow 4Rr^3+r^4>0\to\text{true}\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore \forall x,y,z>0, \sum_{cyc}\frac{x}{y\left(5(x^2+y^2)+x(6y+11z)\right)}\ge\frac{1}{\left(\sum_{cyc}x\right)^2}$$

$$\Rightarrow \sum_{cyc}\frac{m_a}{m_b\left(5(m_a^2+m_b^2)+m_a(6m_b+11m_c)\right)}\ge\frac{1}{\left(\sum_{cyc}m_a\right)^2}\ge\frac{1}{3\sum_{cyc}m_a^2}$$

$$=\frac{4}{9\sum_{cyc}a^2}\overset{\text{Leibnitz}}{\ge}\frac{4}{81R^2}\therefore \text{ in any } \Delta\, ABC,$$

$$\sum_{cyc}\frac{m_a}{m_b\left(5(m_a^2+m_b^2)+m_a(6m_b+11m_c)\right)}$$

$$\ge\frac{4}{81R^2}, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x,y,z>0$. By CBS inequality, we have

$$\sum_{cyc}\frac{x}{y[5(x^2+y^2)+x(6y+11z)]}\ge\frac{(x+y+z)^2}{\sum_{cyc}xy[5(x^2+y^2)+x(6y+11z)]}$$

$$=\frac{(x+y+z)^2}{3\left(\sum_{cyc}xy\right)^2+5\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+3\sum_{cyc}x^2y^2}$$

$$\ge\frac{(x+y+z)^2}{2\left(\sum_{cyc}x^2\right)^2+6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+\left(\sum_{cyc}x^2\right)^2}=\frac{1}{3(x^2+y^2+z^2)}.$$

Then by using Leibniz's inequality, we get

$$\sum_{cyc}\frac{m_a}{m_b[5(m_a{}^2+m_b{}^2)+m_a(6m_b+11m_c)]}\ge\frac{1}{3(m_a{}^2+m_b{}^2+m_c{}^2)}$$

$$=\frac{4}{9(a^2+b^2+c^2)}\ge\frac{4}{81R^2},$$

as desired. Equality holds iff ΔABC is equilateral.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\sum_{cyc}\frac{m_a}{m_b[5(m_a{}^2+m_b{}^2)+m_a(6m_b+11m_c)]}\overset{CBS}{\ge}\frac{\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b}}\right)^2}{\sum_{cyc}[5(m_a{}^2+m_b{}^2)+m_a(6m_b+11m_c)]}$$

$$\overset{AM-GM}{\ge}\frac{3^2}{10\sum_{cyc}m_a{}^2+17\sum_{cyc}m_bm_c}\ge\frac{9}{27\sum_{cyc}m_a{}^2}=\frac{4}{9\sum_{cyc}a^2}\overset{\text{Leibniz}}{\ge}\frac{4}{81R^2},$$

as desired. Equality holds iff ΔABC is equilateral.

1442. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{r_a}{r_b\left(6(r_a^3+r_b^3)+10r_ar_b(r_a+r_b)+r_ar_c(19r_a+30r_b)\right)}\ge\left(\frac{2}{9R}\right)^3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0,\sum_{cyc}\frac{x}{y(6(x^3+y^3)+10xy(x+y)+xz(19x+30y))}$$

$$=\sum_{cyc}\frac{x^2}{6x^4y+6xy^4+10x^3y^2+10x^2y^3+xyz(19x^2+30xy)}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{\left(\sum_{cyc}x\right)^2}{6\sum_{cyc}x^4y+6\sum_{cyc}xy^4+10\sum_{cyc}x^3y^2+10\sum_{cyc}x^2y^3+xyz\left(19\sum_{cyc}x^2+30\sum_{cyc}xy\right)}$$

$$\overset{?}{\ge}\frac{1}{\left(\sum_{cyc}x\right)^3}\Leftrightarrow\left(\sum_{cyc}x\right)^5\overset{?}{\ge}6\sum_{cyc}\left(xy\left(\sum_{cyc}x^3-z^3\right)\right)$$

$$+10\sum_{cyc}\left(x^2y^2\left(\sum_{cyc}x-z\right)\right)+19xyz\sum_{cyc}x^2+30xyz\sum_{cyc}xy$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^5\overset{?}{\ge}6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3\right)-6xyz\left(\sum_{cyc}x^2\right)+10\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)$$

$$-10xyz\left(\sum_{cyc}xy\right)+19xyz\sum_{cyc}x^2+30xyz\sum_{cyc}xy$$

$$=6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3\right)+10\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)+13xyz\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)$$

$$-6xyz\left(\sum_{cyc}xy\right)$$

$$\Leftrightarrow\boxed{\begin{gathered}\left(\sum_{cyc}x\right)^5\underset{(*)}{\overset{?}{\ge}}6\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3\right)+10\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2y^2\right)\\+13xyz\left(\sum_{cyc}x\right)^2-6xyz\left(\sum_{cyc}xy\right)\end{gathered}}$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ *form* sides of a triangle with semiperimeter, circumradius and inradius = s, R, r (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (1) \Rightarrow x = s - X, y = s - Y,$$

$$z = s - Z \text{ and such substitutions} \Rightarrow \sum_{cyc} xy = \sum_{cyc} (s - X)(s - Y)$$

$$\Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (2) \text{ and } \sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3\prod_{cyc}(y + z)$$

$$\overset{via\,(1)}{=} s^3 - 3XYZ = s^3 - 12Rrs \Rightarrow \sum_{cyc} x^3 = s^3 - 12Rrs \to (3) \text{ and also,}$$

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{via\,(1)\,and\,(2)}{=} (4Rr + r^2)^2$$

$$-2\left(\prod_{cyc}(s - X)\right).s = (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 = r^2((4R + r)^2 - 2s^2)$$

$$\to (4) \therefore via\,(1),(2),(3),(4),(*) \Leftrightarrow s^5 \geq 6(4Rr + r^2)(s^3 - 12Rrs)$$

$$+10sr^2((4R + r)^2 - 2s^2) + 13\left(\prod_{cyc}(s - X)\right)s^2 - 6r^2s(4Rr + r^2)$$

$$\Leftrightarrow s^4 \geq 6(4Rr + r^2)(s^2 - 12Rr) + 10r^2((4R + r)^2 - 2s^2) + 13sr^2.s$$

$$-6r^2(4Rr + r^2) \Leftrightarrow s^4 - (24Rr - r^2)s^2 + 4r^2(32R^2 + 4Rr - r^2) \geq 0$$

$$\Leftrightarrow s^2 \geq \frac{24Rr - r^2 + \sqrt{(24Rr - r^2)^2 - 16r^2(32R^2 + 4Rr - r^2)}}{2}$$

$$\Leftrightarrow \boxed{s^2 \overset{(**)}{\geq} \frac{24Rr - r^2 + r * \sqrt{64R^2 - 112Rr + 17r^2}}{2}}$$

$$\text{Now, } 2s^2 \overset{Rouche}{\geq} 2\left(2R^2 + 10Rr - r^2 - 2(R - 2r) * \sqrt{R^2 - 2Rr}\right) \overset{?}{\geq} 24Rr - r^2$$

$$+r * \sqrt{64R^2 - 112Rr + 17r^2}$$

$$\Leftrightarrow \boxed{4R^2 - 4Rr - r^2 - 4(R - 2r) * \sqrt{R^2 - 2Rr} \underbrace{\overset{?}{\geq}}_{(***)} r * \sqrt{64R^2 - 112Rr + 17r^2}}$$

$$\because (4R^2 - 4Rr - r^2)^2 - 16(R^2 - 2Rr)(R - 2r)^2$$

$$= r(64R^3 - 184R^2r + 136Rr^2 + r^3)$$

$$= (R - 2r)(36R^2 + 28R(R - 2r) + 24r^2) + 49r^3 \overset{Euler}{\geq} 49r^3 > 0$$

$$\therefore 4R^2 - 4Rr - r^2 > 4(R - 2r) * \sqrt{R^2 - 2Rr}$$

$$\Rightarrow 4R^2 - 4Rr - r^2 - 4(R - 2r) * \sqrt{R^2 - 2Rr} > 0 \Rightarrow (***) \Leftrightarrow$$

$$\left(4R^2 - 4Rr - r^2 - 4(R - 2r) * \sqrt{R^2 - 2Rr}\right)^2 \geq r^2(64R^2 - 112Rr + 17r^2)$$

$$\Leftrightarrow (4R^2 - 4Rr - r^2)^2 + 16(R^2 - 2Rr)(R - 2r)^2 - r^2(64R^2 - 112Rr + 17r^2)$$

$$\geq 8(R - 2r) * \sqrt{R^2 - 2Rr} * (4R^2 - 4Rr - r^2)$$

$$\Leftrightarrow \boxed{(R-2r)(4R^3-8R^2r+Rr^2+r^3) \ge (R-2r)*\sqrt{R^2-2Rr}*(4R^2-4Rr-r^2)}$$

$$\Leftrightarrow 4R^3-8R^2r+Rr^2+r^3 > \sqrt{R^2-2Rr}*(4R^2-4Rr-r^2)\ \left(\because R-2r \overset{Euler}{\ge} 0\right)$$

$$\Leftrightarrow (4R^3-8R^2r+Rr^2+r^3)^2 > (R^2-2Rr)(4R^2-4Rr-r^2)^2 \Leftrightarrow \boxed{r^5(4R+r)>0}$$

$$\to \text{true} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \forall\, x,y,z>0,$$

$$\sum_{cyc}\frac{x}{y(6(x^3+y^3)+10xy(x+y)+xz(19x+30y))} \ge \frac{1}{(\sum_{cyc} x)^3}$$

$$\Rightarrow \sum_{cyc}\frac{r_a}{r_b\left(6(r_a^3+r_b^3)+10r_ar_b(r_a+r_b)+r_ar_c(19r_a+30r_b)\right)} \ge \frac{1}{(\sum_{cyc} r_a)^3}$$

$$= \frac{1}{(4R+r)^3} \overset{Euler}{\ge} \frac{1}{\left(4R+\frac{R}{2}\right)^3} = \left(\frac{2}{9R}\right)^3, '' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$LHS \overset{CBS}{\ge} \frac{\left(\sum_{cyc}\sqrt{\frac{r_a}{r_b}}\right)^2}{\sum_{cyc}[6(r_a{}^3+r_b{}^3)+10r_ar_b(r_a+r_b)+r_ar_c(19r_a+30r_b)]}$$

$$\overset{AM-GM}{\ge} \frac{3^2}{12\sum_{cyc} r_a{}^3+10\sum_{cyc} r_a{}^2r_b+29\sum_{cyc} r_ar_b{}^2+90r_ar_br_c}$$

$$\overset{AM-GM}{\ge} \frac{9}{12\sum_{cyc} r_a{}^3+29\sum_{cyc} r_ar_b(r_a+r_b)+33r_ar_br_c}$$

$$= \frac{9}{12(\sum_{cyc} r_a)^3-7\sum_{cyc} r_a.\sum_{cyc} r_br_c-18r_ar_br_c}$$

$$= \frac{9}{12(4R+r)^3-7(4R+r)s^2-18s^2r}$$

$$\overset{Gerretsen}{\ge} \frac{9}{12(4R+r)^3-(28R+25r)(16Rr-5r^2)} \overset{?}{\ge} \left(\frac{2}{9R}\right)^3$$

$$\Leftrightarrow 417R^3-1024R^2r+928Rr^2-1096r^3 \ge 0$$

$$\Leftrightarrow (R-2r)(417R^2-190Rr+548r^2) \ge 0$$

which is true by Euler's inequality $R \ge 2r$. Equality holds iff ΔABC is equilateral.

1443. **In any Δ ABC, the following relationship holds :**

$$\frac{(r_aw_b)^5}{r_a^5+w_b^5}+\frac{(r_bw_c)^5}{r_b^5+w_c^5}+\frac{(r_cw_a)^5}{r_c^5+w_a^5} \le \left(\frac{3}{2}\right)^6\frac{(81R^5-2560r^5)^2}{32r^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$w_a = \frac{2\sqrt{bc}}{b+c}.\sqrt{s(s-a)} \overset{A-G}{\leq} \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-b)(s-c)}} \overset{G-H}{\leq}$$

$$\frac{rs(s-a)(s-b+s-c)}{2(s-a)(s-b)(s-c)} = \frac{rsa(s-a)}{2r^2s} = \frac{a(s-a)}{2r}$$

$$\Rightarrow w_a^5 \leq s^2(s-a)^2.\frac{a(s-a)}{2r} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} w_a^5 \leq \frac{s^2}{2r}.\sum_{cyc} a(s-a)^3 = \frac{s^2}{2r}.\sum_{cyc} a(s^3-3s^2a+3sa^2-a^3)$$

$$= \frac{s^2}{2r}.\begin{pmatrix} s^3(2s)-6s^2(s^2-4Rr-r^2)+6s^2(s^2-6Rr-3r^2) \\ +16r^2s^2-2\left((s^2+4Rr+r^2)^2-16Rrs^2\right) \end{pmatrix}$$

$$= \frac{s^2}{2r}.\left(4Rrs^2-2r^2(4R+r)^2\right) \overset{Gerretsen}{\leq}$$

$$\frac{s^2}{2r}.\left(4Rr(4R^2+4Rr+3r^2)-2r^2(4R+r)^2\right)$$

$$\Rightarrow \sum_{cyc} w_a^5 \overset{(i)}{\leq} s^2(8R^3-8R^2r-2Rr^2-r^3)$$

$$r_a \leq \frac{a^2}{4r} \Rightarrow r_a^5 \leq s^4\tan^4\frac{A}{2}.\frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{4r} = \frac{4R^2s^4}{r}\left(\tan^2\frac{A}{2}\right)\left(1-\cos^2\frac{A}{2}\right)^2$$

$$= \frac{4R^2s^4}{r}\left(\tan^2\frac{A}{2}\right)\left(1+\cos^4\frac{A}{2}-2\cos^2\frac{A}{2}\right) = \frac{4R^2s^4}{r}\left(\frac{r_a^2}{s^2}+\frac{\sin^2 A}{4}-(1-\cos A)\right)$$

$$\Rightarrow r_a^5 \leq \frac{4R^2s^4}{r}\left(\frac{r_a^2}{s^2}+\frac{a^2}{16R^2}-1+\cos A\right) \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} r_a^5 \leq \frac{4R^2s^4}{r}\left(\frac{(4R+r)^2-2s^2}{s^2}+\frac{\sum_{cyc}a^2}{16R^2}-3+1+\frac{r}{R}\right)$$

$$\overset{Leibnitz}{\leq} \frac{4R^2s^4}{r}\left(\frac{(4R+r)^2-4s^2}{s^2}+\frac{9}{16}+\frac{r}{R}\right)$$

$$= \frac{Rs^2}{4r}.\left(16R(4R+r)^2-(55R-16r)s^2\right) \overset{Gerretsen}{\leq}$$

$$\frac{Rs^2}{4r}.\left(16R(4R+r)^2-(55R-16r)(16Rr-5r^2)\right)$$

$$\Rightarrow \sum_{cyc} r_a^5 \overset{(ii)}{\leq} \frac{Rs^2}{4r}.\left(256R^3-752R^2r+547Rr^2-80r^3\right)$$

$$\because 256R^3-752R^2r+547Rr^2-80r^3$$

$$= (R-2r)\left(136R^2+120R(R-2r)+67r^2\right)+54r^3 \overset{Euler}{\geq} 54r^3 > 0 \text{ and}$$

$$8R^3-8R^2r-2Rr^2-r^3 = (R-2r)\left(8R^2+8Rr+14r^2\right)+27r^3 \overset{Euler}{\geq} 27r^3 > 0$$

$$\therefore \frac{(r_a w_b)^5}{r_a^5+w_b^5}+\frac{(r_b w_c)^5}{r_b^5+w_c^5}+\frac{(r_c w_a)^5}{r_c^5+w_a^5} \overset{A-H}{\le} \frac{1}{4}\left(\sum_{cyc} w_a^5+\sum_{cyc} r_a^5\right)$$

$$\overset{via\ (i),(ii)}{\le} \frac{s^2}{4}(8R^3-8R^2r-2Rr^2-r^3)+\frac{Rs^2}{16r}.(256R^3-752R^2r+547Rr^2-80r^3)$$

$$\overset{Mitrinovic}{\le} \frac{27R^2}{16}(8R^3-8R^2r-2Rr^2-r^3)$$

$$+\frac{27R^3}{64r}.(256R^3-752R^2r+547Rr^2-80r^3) \overset{?}{\le} \left(\frac{3}{2}\right)^6 \frac{(81R^5-2560r^5)^2}{32r^5}$$

$$\Leftrightarrow 27(81R^5-2560r^5)^2 \overset{?}{\ge} 32r^4\begin{pmatrix} 4R^2r(8R^3-8R^2r-2Rr^2-r^3) \\ +R^3(256R^3-752R^2r+547Rr^2-80r^3)\end{pmatrix}$$

$$\Leftrightarrow 177147t^{10}-8192t^6-11174400t^5-16480t^4+2816t^3+128t^2$$

$$+176947200 \overset{?}{\ge} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(\begin{matrix}(t-2)* \\ \begin{pmatrix}177147t^8+708588t^7+2125764t^6 \\ +5668704t^5+14163568t^4+22805056t^3 \\ +34549472t^2+46980480t+49724160\end{pmatrix} \\ +10974720\end{matrix}\right) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{Euler}{\ge} 2 \Rightarrow \frac{(r_a w_b)^5}{r_a^5+w_b^5}+\frac{(r_b w_c)^5}{r_b^5+w_c^5}+\frac{(r_c w_a)^5}{r_c^5+w_a^5}$$

$$\le \left(\frac{3}{2}\right)^6 \frac{(81R^5-2560r^5)^2}{32r^5}\ \forall \Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1444. Let ABC be an acute triangle and let H, I be the orthocenter, incenter of ΔABC respectively and $A_1B_1C_1$ the orthic triangle. Then

$$\frac{HA_1.HB_1.HC_1}{IA_1.IB_1.IC_1}.\frac{HA.HB.HC}{IA.IB.IC} \le \frac{R^5}{32r^5}$$

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $HA = 2R\cos A$ and $HA_1 = 2R\cos B\cos C$ (and analogs), *and since*

$$\cos A\cos B\cos C \overset{AM-GM}{\le} \left(\frac{\cos A+\cos B+\cos C}{3}\right)^3 \overset{Jensen}{\le} \cos^3\frac{\pi}{3}=\frac{1}{8}, \text{ then}$$

$$HA.HB.HC = 8R^3\cos A\cos B\cos C \le R^3 \text{ and}$$

$$HA_1.HB_1.HC_1 = 8R^3(\cos A\cos B\cos C)^2 = \frac{R^3}{8}.$$

Also, we have $IA.IB.IC = \frac{r}{\sin\frac{A}{2}}.\frac{r}{\sin\frac{B}{2}}.\frac{r}{\sin\frac{C}{2}} = \frac{r^3}{\frac{r}{4R}} = 4Rr^2.$

Now, since $IA_1 \ge distance(I,(BC)) = r$ (and analogs), then $IA_1.IB_1.IC_1 \ge r^3.$

Therefore

$$\frac{HA_1.HB_1.HC_1}{IA_1.IB_1.IC_1}\cdot\frac{HA.HB.HC}{IA.IB.IC}\le\frac{\frac{R^3}{8}}{r^3}\cdot\frac{R^3}{4Rr^2}=\frac{R^5}{32r^5}.$$

Equality holds iff ΔABC is equilateral.

1345.

In any ΔABC and $n,m\in\mathbb{N}$ such that $n\ge m-2$ the following relationship holds :

$$\frac{r_a^{\,n}(r_a^{\,2}+r_br_c)}{(r_b+r_c)^m}+\frac{r_b^{\,n}(r_b^{\,2}+r_cr_a)}{(r_c+r_a)^m}+\frac{r_c^{\,n}(r_c^{\,2}+r_ar_b)}{(r_a+r_b)^m}\ge\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{r_a^{\,n}(r_a^{\,2}+r_br_c)}{(r_b+r_c)^m}=\frac{r_a^{\,n}}{(r_b+r_c)^m}(r_a-r_b)(r_a-r_c)+\frac{r_a^{\,n+1}}{(r_b+r_c)^{m-1}}\text{ (and analogs)}$$

WLOG, we may assume that $r_a\ge r_b\ge r_c$. We have

$$\frac{r_a^{\,n}}{(r_b+r_c)^m}\ge\frac{r_b^{\,n}}{(r_c+r_a)^m}\ge\frac{r_c^{\,n}}{(r_a+r_b)^m},$$

then by the Generalized Schur inequality, we have

$$\sum_{cyc}\frac{r_a^{\,n}}{(r_b+r_c)^m}(r_a-r_b)(r_a-r_c)\ge 0.$$

Therefore

$$\sum_{cyc}\frac{r_a^{\,n}(r_a^{\,2}+r_br_c)}{(r_b+r_c)^m}\ge\sum_{cyc}\frac{r_a^{\,n+1}}{(r_b+r_c)^{m-1}}\overset{Chebyshev}{\ge}\frac{1}{3}\cdot\sum_{cyc}r_a^{\,n+1}\cdot\sum_{cyc}\frac{1}{(r_b+r_c)^{m-1}}$$

$$\overset{H\ddot{o}lder}{\ge}\frac{1}{3}\cdot\frac{(r_a+r_b+r_c)^{n+1}}{3^n}\cdot\frac{3^m}{2^{m-1}(r_a+r_b+r_c)^{m-1}}=\frac{(4R+r)^{n-m+2}}{3^{n+1-m}.2^{m-1}}$$

$$\overset{Euler}{\ge}\frac{(9r)^{n-m+2}}{3^{n+1-m}.2^{m-1}}=\frac{3^{n-m+3}.r^{n-m+2}}{2^{m-1}}.$$

as desired. Equality holds if and only if ΔABC is equilateral.

1446. In any acute Δ ABC, the following identity occurs :

$$\sin\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)-\sin\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)\sum_{cyc}AH\cos\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)$$

$$=\frac{(a^2+b^2+c^2-6R^2)\cos(\hat{B}-\hat{C})}{R}$$

Proposed by Radu Diaconu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} AH\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right) = R\sum_{cyc} 2\cos\hat{A}\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)$$

$$= R\sum_{cyc}\left(\cos\left(\hat{A}+\hat{B}-\hat{C}-\frac{\pi}{4}\right)+\cos\left(\hat{A}-\hat{B}+\hat{C}+\frac{\pi}{4}\right)\right)$$

$$= R\sum_{cyc}\left(\cos\left(\frac{3\pi}{4}-2\hat{C}\right)+\cos\left(\frac{5\pi}{4}-2\hat{B}\right)\right)$$

$$= R\sum_{cyc}\left(-\frac{1}{\sqrt{2}}\cos\widehat{2C}+\frac{1}{\sqrt{2}}\sin\widehat{2C}-\frac{1}{\sqrt{2}}\cos\widehat{2B}-\frac{1}{\sqrt{2}}\sin\widehat{2B}\right)$$

$$= \frac{R}{\sqrt{2}}\left(\sum_{cyc}\sin\widehat{2C}-\sum_{cyc}\sin\widehat{2B}-\sum_{cyc}\cos\widehat{2C}-\sum_{cyc}\cos\widehat{2B}\right)$$

$$= -R*\sqrt{2}\left(\sum_{cyc}\cos\widehat{2A}\right) \text{ and } \because \sin\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos(\hat{B}-\hat{C})+\sin(\hat{B}-\hat{C})\right)$$

$$\therefore \boxed{\sin\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)\sum_{cyc} AH\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right) \overset{(*)}{=} R\left(\sum_{cyc}\cos\widehat{2A}\right)\left(-\cos(\hat{B}-\hat{C})-\sin(\hat{B}-\hat{C})\right)}$$

$$\text{Also, }\sum_{cyc} AH\cos\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right) = R\sum_{cyc} 2\cos\hat{A}\cos\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)$$

$$= R\sum_{cyc}\left(\cos\left(\hat{A}+\hat{B}-\hat{C}+\frac{\pi}{4}\right)+\cos\left(\hat{A}-\hat{B}+\hat{C}-\frac{\pi}{4}\right)\right)$$

$$= R\sum_{cyc}\left(\cos\left(\frac{5\pi}{4}-2\hat{C}\right)+\cos\left(\frac{3\pi}{4}-2\hat{B}\right)\right)$$

$$= R\sum_{cyc}\left(-\frac{1}{\sqrt{2}}\cos\widehat{2C}-\frac{1}{\sqrt{2}}\sin\widehat{2C}-\frac{1}{\sqrt{2}}\cos\widehat{2B}+\frac{1}{\sqrt{2}}\sin\widehat{2B}\right)$$

$$= \frac{R}{\sqrt{2}}\left(-\sum_{cyc}\cos\widehat{2C}-\sum_{cyc}\sin\widehat{2C}-\sum_{cyc}\cos\widehat{2B}+\sum_{cyc}\sin\widehat{2B}\right)$$

$$= -R*\sqrt{2}\left(\sum_{cyc}\cos\widehat{2A}\right) \text{ and } \because -\sin\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos(\hat{B}-\hat{C})-\sin(\hat{B}-\hat{C})\right)$$

$$\therefore \boxed{-\sin\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)\sum_{cyc} AH\cos\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right) \overset{(**)}{=} R\left(\sum_{cyc}\cos\widehat{2A}\right)\left(\sin(\hat{B}-\hat{C})-\cos(\hat{B}-\hat{C})\right)}$$

$$\therefore (*),(**) \Rightarrow \sin\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right)\sum_{cyc} AH\cos\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)$$

$$-\sin\left(\hat{B}-\hat{C}-\frac{\pi}{4}\right)\sum_{cyc} AH\cos\left(\hat{B}-\hat{C}+\frac{\pi}{4}\right) = -2R\left(-1-4\prod_{cyc}\cos\hat{A}\right)\cos(\hat{B}-\hat{C})$$

$$= 2R\cos(\hat{B}-\hat{C}) * \left(1+\frac{s^2-4R^2-4Rr-r^2}{R^2}\right)$$

$$= 2R\cos(\hat{B}-\hat{C}) * \left(\frac{s^2-4Rr-r^2-3R^2}{R^2}\right)$$

$$= \frac{\cos(\hat{B}-\hat{C})}{R} * \left(2(s^2-4Rr-r^2)-6R^2\right)$$

$$= \frac{(a^2+b^2+c^2-6R^2)\cos(\hat{B}-\hat{C})}{R} \text{ (QED)}$$

1447. In any $\Delta\, ABC, \Delta\, A'B'C'$, the following relationship holds :

$$\min\left\{\sum_{cyc}\frac{m_a'}{m_b'+m_c'},\sum_{cyc}\frac{a}{b+c}\right\}+\frac{R'R^2}{r'r^2} \geq 8+\max\left\{\sum_{cyc}\frac{m_a}{m_b+m_c},\sum_{cyc}\frac{a'}{b'+c'}\right\}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum m_a m_b \overset{\text{Tereshin}}{\geq} \sum \frac{(b^2+c^2)(c^2+a^2)}{16R^2} = \frac{3\sum_{cyc}a^2b^2+\sum_{cyc}a^4}{16R^2}$$

$$\geq \frac{3\sum_{cyc}a^2b^2+\sum_{cyc}a^2b^2}{16R^2} = \sum_{cyc}\frac{a^2b^2}{4R^2} \Rightarrow \frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}$$

$$\leq \frac{4R^2 * \frac{3}{2}(s^2-4Rr-r^2)}{\sum_{cyc}a^2b^2} \overset{?}{\leq} \frac{R}{2r} \Leftrightarrow \sum_{cyc}a^2b^2 \overset{?}{\geq} 12Rr(s^2-4Rr-r^2)$$

$$\Leftrightarrow (s^2+4Rr+r^2)^2-16Rrs^2 \overset{?}{\geq} 12Rr(s^2-4Rr-r^2)$$

$$\Leftrightarrow s^4-(20Rr-2r^2)s^2+r^2(64R^2+20Rr+r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow s^2 \geq \frac{20Rr-2r^2+\sqrt{(20Rr-2r^2)^2-4r^2(64R^2+20Rr+r^2)}}{2}$$

$$\Leftrightarrow s^2 \geq \frac{20Rr-2r^2+4r*\sqrt{9R^2-10Rr}}{2} \Leftrightarrow s^2 \overset{(*)}{\geq} 10Rr-r^2+2r*\sqrt{9R^2-10Rr}$$

$$\text{Now, } s^2 \overset{\text{Rouche}}{\geq} 2R^2+10Rr-r^2-2(R-2r)*\sqrt{R^2-2Rr} \overset{?}{\geq} 10Rr-r^2+$$

$$2r*\sqrt{9R^2-10Rr} \Leftrightarrow R^2-(R-2r)*\sqrt{R^2-2Rr} \underbrace{\overset{?}{\geq}}_{(**)} r*\sqrt{9R^2-10Rr}$$

$$\because R^4-(R^2-2Rr)(R-2r)^2 = 2Rr(3R^2-6Rr+4r^2) \overset{\text{Euler}}{\geq} 8Rr^3 > 0$$

$$\Rightarrow R^2 > (R-2r)*\sqrt{R^2-2Rr} \Rightarrow R^2-(R-2r)*\sqrt{R^2-2Rr} > 0$$

$$\Rightarrow (**) \Leftrightarrow \left(R^2-(R-2r)*\sqrt{R^2-2Rr}\right)^2 \geq r^2(9R^2-10Rr)$$

$$\Leftrightarrow R^4+(R^2-2Rr)(R-2r)^2-r^2(9R^2-10Rr) \geq 2R^2*(R-2r)*\sqrt{R^2-2Rr}$$

$$\Leftrightarrow R(R-2r)(2R^2-2Rr-r^2) \geq 2R^2*(R-2r)*\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 2R^2-2Rr-r^2 > 2R*\sqrt{R^2-2Rr}\ \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)$$

$$\Leftrightarrow (2R^2-2Rr-r^2)^2-4R^2(R^2-2Rr) > 0$$

$$\Leftrightarrow r^3(4R+r) > 0 \rightarrow true \Rightarrow (**) \Rightarrow (*) \text{ is true} \Rightarrow \frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a} \leq \frac{R}{2r}$$

$$\Rightarrow \frac{3}{2}+\frac{R'R^2}{r'r^2} \overset{\text{Euler}}{\geq} \frac{3}{2}+8*\frac{R^2}{4r^2} \geq \frac{3}{2}+8\left(\frac{\sum_{cyc} m_a^2}{\sum_{cyc} m_am_b}\right)^2 \therefore \text{in order to prove:}$$

$$\frac{3}{2}+\frac{R'R^2}{r'r^2} \geq 8+\sum_{cyc}\frac{m_a}{m_b+m_c}, \text{it suffices to prove:}$$

$$\frac{3}{2}+8\left(\frac{\sum_{cyc} m_a^2}{\sum_{cyc} m_am_b}\right)^2 \overset{(\blacksquare)}{\geq} 8+\sum_{cyc}\frac{m_a}{m_b+m_c}$$

$$\text{Now, }\frac{3}{2}+8\left(\frac{\sum_{cyc} a^2}{\sum_{cyc} ab}\right)^2 \overset{?}{\geq} 8+\sum_{cyc}\frac{a}{b+c}$$

$$\Leftrightarrow \frac{8}{(s^2+4Rr+r^2)^2}\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right) \overset{?}{\geq} 2s\sum_{cyc}\frac{1}{b+c}-\frac{9}{2}$$

$$\Leftrightarrow \frac{8(3s^2-4Rr-r^2)(s^2-12Rr-3r^2)}{(s^2+4Rr+r^2)^2} \overset{?}{\geq} \frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{9}{2}$$

$$= \frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)} \Leftrightarrow 8(3s^2-4Rr-r^2)(2s^2-24Rr-6r^2)(s^2+2Rr+r^2)$$

$$\underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}} (s^2-10Rr-7r^2)(s^2+4Rr+r^2)^2$$

$$2s^2-24Rr-6r^2 \overset{?}{\geq} s^2-10Rr-7r^2 \Leftrightarrow s^2-16Rr+5r^2+2r(R-2r) \overset{?}{\geq} 0$$

$\rightarrow$ true via Gerretsen and Euler $\therefore$ in order to prove $(\blacksquare\blacksquare)$, it suffices to prove:

$$8(3s^2-4Rr-r^2)(s^2+2Rr+r^2) > (s^2+4Rr+r^2)^2$$

$$\Leftrightarrow 23s^4+(8Rr+14r^2)s^2-r^2(80R^2+56Rr+9r^2) \overset{(\blacksquare\blacksquare\blacksquare)}{>} 0$$

$$\text{Now, LHS of }(\blacksquare\blacksquare\blacksquare) \overset{\text{Gerretsen}}{\geq} \left(23(16Rr-5r^2)+(8Rr+14r^2)\right)(16Rr-5r^2)$$

$$-r^2(80R^2+56Rr+9r^2) = r^2\left((R-2r)(5936R+8320r)+17136r^2\right) \overset{\text{Euler}}{\geq}$$

$$17136r^4 > 0 \Rightarrow (\blacksquare\blacksquare\blacksquare) \Rightarrow (\blacksquare\blacksquare) \text{ is true} \Rightarrow \frac{3}{2}+8\left(\frac{\sum_{cyc} a^2}{\sum_{cyc} ab}\right)^2 \geq 8+\sum_{cyc}\frac{a}{b+c}$$

and implementing it on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, we arrive at:

$$\frac{3}{2}+8\left(\frac{\sum_{cyc} m_a^2}{\sum_{cyc} m_a m_b}\right)^2 \geq 8+\sum_{cyc}\frac{m_a}{m_b+m_c} \Rightarrow (\blacksquare) \text{ is true}$$

$$\Rightarrow \boxed{\frac{3}{2}+\frac{R'R^2}{r'r^2} \overset{(\text{?})}{\geq} 8+\sum_{cyc}\frac{m_a}{m_b+m_c}}$$

$$\text{Again}, \frac{3}{2}+\frac{4R}{r} \overset{?}{\geq} 8+\sum_{cyc}\frac{a}{b+c} \Leftrightarrow \frac{4R}{r}-8 \overset{?}{\geq} 2s\sum_{cyc}\frac{1}{b+c}-\frac{9}{2}$$

$$= \frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{9}{2} = \frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)} \Leftrightarrow \frac{4(R-2r)}{r} \overset{?}{\geq} \frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)}$$

$$\Leftrightarrow (8R-17r)s^2+r(16R^2-14Rr-9r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow (8R-16r)s^2-rs^2+r(16R^2-14Rr-9r^2) \underset{(\bullet)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\geq} (8R-16r)(16Rr-5r^2)-r(4R^2+4Rr+3r^2)$$

$$+r(16R^2-14Rr-9r^2) \overset{?}{\geq} 0 \Leftrightarrow 70R^2-157Rr+34r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(70R-17r) \overset{?}{\geq} 0 \to \text{true via Euler} \Rightarrow (\bullet) \text{ is true} \Rightarrow \frac{3}{2}+\frac{4R}{r}$$

$$\geq 8+\sum_{cyc}\frac{a}{b+c} \therefore \frac{3}{2}+\frac{R'R^2}{r'r^2} \overset{\text{Euler}}{\geq} \frac{3}{2}+\frac{4R'}{r'} \geq 8+\sum_{cyc}\frac{a'}{b'+c'}$$

$$\therefore \boxed{\frac{3}{2}+\frac{R'R^2}{r'r^2} \overset{(\text{??})}{\geq} 8+\sum_{cyc}\frac{a'}{b'+c'}}$$

$$\therefore \min\left\{\sum_{cyc}\frac{m_a'}{m_b'+m_c'}, \sum_{cyc}\frac{a}{b+c}\right\}+\frac{R'R^2}{r'r^2} \overset{\text{Nesbitt}}{\geq} \frac{3}{2}+\frac{R'R^2}{r'r^2} \overset{\text{via (?),(??)}}{\geq}$$

$$8+\max\left\{\sum_{cyc}\frac{m_a}{m_b+m_c}, \sum_{cyc}\frac{a'}{b'+c'}\right\}, \text{"="} \text{ iff } \Delta ABC, \Delta A'B'C' \text{ are both equilateral (QED)}$$

1448. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq \frac{9}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1+3x^2} \overset{?}{\geq} 1-\frac{\sqrt{3}x}{2} \Leftrightarrow \frac{\sqrt{3}x}{2} \overset{?}{\geq} 1-\frac{1}{1+3x^2} = \frac{3x^2}{1+3x^2} \Leftrightarrow 1+3x^2 \overset{?}{\geq} 2\sqrt{3}x$$

$$\Leftrightarrow \left(1-\sqrt{3}x\right)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \frac{1}{1+3x^2} \geq 1 - \frac{\sqrt{3}x}{2}$$

$$\Rightarrow \frac{1}{1+3\tan^2\frac{A}{2}} - \left(1 - \frac{\sqrt{3}\tan\frac{A}{2}}{2}\right) \geq 0$$

$$\Rightarrow \left(2+\sqrt{3}\tan\frac{B}{2}\right)\left(\frac{1}{1+3\tan^2\frac{A}{2}} - \left(1 - \frac{\sqrt{3}\tan\frac{A}{2}}{2}\right)\right) \geq 0$$

$$\Rightarrow \frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq \left(2+\sqrt{3}\tan\frac{B}{2}\right)\left(1 - \frac{\sqrt{3}\tan\frac{A}{2}}{2}\right)$$

$$= 2+\sqrt{3}\tan\frac{B}{2} - \sqrt{3}\tan\frac{A}{2} - \frac{3}{2}\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\therefore \frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq 2+\sqrt{3}\tan\frac{B}{2} - \sqrt{3}\tan\frac{A}{2} - \frac{3r_ar_b}{2s^2} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq 6+\sqrt{3}\sum_{cyc}\tan\frac{B}{2} - \sqrt{3}\sum_{cyc}\tan\frac{A}{2} - \frac{3}{2s^2}\sum_{cyc}r_ar_b = 6 - \frac{3s^2}{2s^2} = \frac{9}{2}$$

$$\therefore \text{in any } \Delta\, ABC, \sum_{cyc}\frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq \frac{9}{2}, \text{"="} \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1449.

In any Δ ABC with I $\to$ incenter, the following relationship holds :

$$\sqrt{\frac{2R}{r}} \cdot \sum_{cyc}\frac{n_a}{\sqrt{(b-c)^2+4r^2}} \geq \frac{1}{\sqrt{2}} \cdot \sum_{cyc}\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}} + \frac{1}{r}\sum_{cyc} AI$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \overset{?}{\geq} 32Rr\cos^2\frac{A}{2} \overset{\text{via (i)}}{=} 8r(r_b+r_c) = 8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \overset{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore b+c\ge\sqrt{32Rr}.\cos\frac{A}{2}\Rightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\ge\sqrt{32Rr}.\cos\frac{A}{2}\Rightarrow\cos\frac{B-C}{2}\overset{(ii)}{\ge}\sqrt{\frac{2r}{R}}$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$

$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}\Rightarrow a^2n_a^2=a^2s^2-sa(a^2-(b-c)^2)$$

$$\Rightarrow a^2n_a^2=a^2s^2-sa^3+sa(b-c)^2\overset{?}{=}4r^2s^2+s^2(b-c)^2$$

$$\Leftrightarrow a^2s^2-sa^3+sa(b-c)^2\overset{?}{=}s(s-a)(a^2-(b-c)^2)+s^2(b-c)^2$$

$$=a^2s^2-sa^3-s(s-a)(b-c)^2+s^2(b-c)^2$$

$$=a^2s^2-sa^3-s^2(b-c)^2+sa(b-c)^2+s^2(b-c)^2$$

$$\Leftrightarrow a^2s^2-sa^3+sa(b-c)^2\overset{?}{=}a^2s^2-sa^3+sa(b-c)^2\to\text{true}$$

$$\therefore\frac{n_a^2}{(b-c)^2+4r^2}=\frac{s^2}{a^2}\Rightarrow\sqrt{\frac{2R}{r}}.\frac{n_a}{\sqrt{(b-c)^2+4r^2}}=\sqrt{\frac{R}{2r}}.\frac{2s}{a}$$

$$=\sqrt{\frac{R}{2r}}.\frac{4R.2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}=\sqrt{\frac{R}{2r}}.\frac{\sin\frac{A}{2}+\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\sqrt{\frac{R}{2r}}+\sqrt{\frac{R}{2r}}.\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$$

$$\overset{\text{via (ii)}}{\ge}\sqrt{\frac{R}{2r}}+\sqrt{\frac{R}{2r}}.\frac{\sqrt{\frac{2r}{R}}}{\sin\frac{A}{2}}=\sqrt{\frac{R}{2r}}+\frac{AI}{r}\ge\sqrt{\frac{\frac{m_b}{h_c}+\frac{m_c}{h_b}}{2}}+\frac{AI}{r}$$

$\left(\because\frac{R}{r}\ge\frac{m_b}{h_c}+\frac{m_c}{h_b}\right.$; reference: article titled "NewTriangle Inequalities With Brocard's Angle" by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 − 7, published at : www.ssmrmh.ro$\left.\right)$

$$\therefore\sqrt{\frac{2R}{r}}.\frac{n_a}{\sqrt{(b-c)^2+4r^2}}\ge\frac{1}{\sqrt2}.\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{AI}{r}\text{ and analogs}$$

$$\Rightarrow\sqrt{\frac{2R}{r}}.\sum_{cyc}\frac{n_a}{\sqrt{(b-c)^2+4r^2}}\ge\frac{1}{\sqrt2}.\sum_{cyc}\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{1}{r}\sum_{cyc}AI,$$

" = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We prove the result by using the following inequality (see, Bogdan Fuştei, Mohamed Amine Ben Ajiba, *New Triangle Inequalities With Brocard's Angle*, Lemma 12, 6 − 7, www.ssmrmh.ro)

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}\le\frac{R}{r}\quad\text{(and analogs)}$$

Also, we have

$$\frac{AI}{r}=\frac{1}{\sin\frac{A}{2}}=\sqrt{\frac{bc}{(s-b)(s-c)}}=\sqrt{\frac{2Rsr.(b+c-a).a}{sr^2.a^2}}\overset{AM-GM}{\leq}\sqrt{\frac{R}{2r}.\frac{(b+c-a)+a}{a}}$$

$$=\sqrt{\frac{R}{2r}.\frac{b+c}{a}}.$$

$$(b-c)^2+4r^2=(b-c)^2+\frac{(s-a).4(s-b)(s-c)}{s}$$

$$=(b-c)^2+\frac{(s-a)[a^2-(b-c)^2]}{s}$$

$$=\frac{a^2(s-a)+a(b-c)^2}{s}=\frac{a^2}{s^2}\left(s(s-a)+\frac{s(b-c)^2}{a}\right)=\left(\frac{an_a}{s}\right)^2.$$

Using these results, we have

$$\frac{1}{\sqrt{2}}.\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{AI}{r}\leq\sqrt{\frac{R}{2r}}+\sqrt{\frac{R}{2r}.\frac{b+c}{a}}=\sqrt{\frac{2R}{r}.\frac{s}{a}}$$

$$=\sqrt{\frac{2R}{r}}.\frac{n_a}{\sqrt{(b-c)^2+4r^2}}\quad\text{(and analogs)}$$

Therefore

$$\frac{1}{\sqrt{2}}\sum_{cyc}\sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{1}{r}.\sum_{cyc}AI\leq\sqrt{\frac{2R}{r}}.\sum_{cyc}\frac{n_a}{\sqrt{(b-c)^2+4r^2}}.$$

Equality holds iff ΔABC is equilateral.

1450. In ΔABC the following relationship holds:

$$\frac{1}{m_a^4m_b}+\frac{1}{m_b^4m_c}+\frac{1}{m_c^4m_a}\geq\frac{32}{81(81R^5-2560r^5)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{1}{m_a^4m_b}=\sum_{cyc}\frac{\left(\frac{1}{m_a}\right)^4}{m_b}\overset{HOLDER}{\geq}\frac{\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^4}{9(m_a+m_b+m_c)}\geq$$

$$\overset{LEUENBERGER}{\geq}\frac{\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)^4}{9(4R+r)}\overset{BERGSTROM}{\geq}\frac{\left(\frac{9}{m_a+m_b+m_c}\right)^4}{9(4R+r)}\geq$$

$$\overset{LEUENBERGER}{\geq}\frac{\left(\frac{9}{4R+r}\right)^4}{9(4R+r)}\overset{EULER}{\geq}\frac{\left(\frac{9}{4R+\frac{R}{2}}\right)^4}{9(4R+r)}=\frac{\left(\frac{2}{R}\right)^4}{9(4R+r)}=\frac{16}{9R^4(4R+r)}$$

Remains to prove:

$$\frac{16}{9R^4(4R+r)}\geq\frac{32}{81(81R^5-2560r^5)}$$

$$\frac{1}{R^4(4R+r)}\geq\frac{2}{9(81R^5-2560r^5)}$$

$$721R^5-2R^4r-9\cdot 2560r^5\geq 0$$

$$721R^4(R-2r)+2^5\cdot 3^2\cdot 5r(R^4-16r^4)\geq 0$$

$$(R-2r)\left(721R^4+2^5\cdot 3^2\cdot 5r(R+2r)(R^2+4r^2)\right)\geq 0$$

$$R-2r\geq 0$$

$$R\geq 2r\text{ (Euler)}$$

Equality holds for: $a=b=c$.

1451. In any Δ ABC, the following relationship holds :

$$\sqrt{1+\frac{R}{2r}\left(\frac{h_a+h_b+h_c}{s}\right)^2}\geq\sqrt{\frac{h_b+h_c}{h_a+h_c}}+\sqrt{\frac{h_a+h_c}{h_b+h_c}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{1+\frac{R}{2r}\left(\frac{h_a+h_b+h_c}{s}\right)^2}\geq\sqrt{\frac{h_b+h_c}{h_a+h_c}}+\sqrt{\frac{h_a+h_c}{h_b+h_c}}$$

$$\Leftrightarrow 1+\frac{R}{2r}\cdot\frac{\left(\sum_{cyc}ab\right)^2}{4R^2s^2}\geq 2+\frac{ca+ab}{bc+ab}+\frac{bc+ab}{ca+ab}$$

$$\Leftrightarrow\frac{(ab+bc+ca)^2}{abc(a+b+c)}\geq\frac{(ca+ab)^2+(bc+ab)^2+(ca+ab)(bc+ab)}{(ca+ab)(bc+ab)}$$

$$\Leftrightarrow (ca+ab)(bc+ab)(ab+bc+ca)^2$$

$$-abc(a+b+c)\left((ca+ab)^2+(bc+ab)^2+(ca+ab)(bc+ab)\right)\geq 0$$

$$\Leftrightarrow a^4b^4 + a^2b^2c^4 - 2a^3b^3c^2 \geq 0 \Leftrightarrow a^2b^2(ab - c^2)^2 \geq 0 \to \text{true}$$

$$\therefore \sqrt{1 + \frac{R}{2r}\left(\frac{h_a + h_b + h_c}{s}\right)^2} \geq \sqrt{\frac{h_b + h_c}{h_a + h_c}} + \sqrt{\frac{h_a + h_c}{h_b + h_c}}\ \forall\, \Delta\, ABC\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : In ΔABC, we have

$$\frac{b}{c} + \frac{c}{b} \leq \frac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{2F} \qquad (*)$$

Proof : Since $16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$, then we have

$$(*) \Leftrightarrow (b^2 + c^2)\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)} \leq 2bc\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

$$\overset{squaring}{\Leftrightarrow} (2b^2c^2 + b^4 + c^4)[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)] \leq 4b^2c^2(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\Leftrightarrow -a^4(b^2 + c^2)^2 + 2(b^4 + c^4)(a^2b^2 + c^2a^2) - (b^4 + c^4)^2 = -[a^2(b^2 + c^2) - (b^4 + c^4)]^2 \leq 0,$$

which is true and the proof of the lemma is complete.

Now, since $\sqrt{h_a + h_b}, \sqrt{h_b + h_c}, \sqrt{h_a + h_c}$ can be the sides of triangle with area F' such that

$$16F'^2 = 2\sum_{cyc}\sqrt{h_a + h_b}^2\sqrt{h_a + h_c}^2 - \sum_{cyc}\sqrt{h_b + h_c}^4 = 4\sum_{cyc} h_b h_c = \frac{8s^2r}{R}.$$

Then by using the lemma in this triangle, we obtain

$$\frac{\sqrt{h_b + h_c}}{\sqrt{h_a + h_c}} + \frac{\sqrt{h_a + h_c}}{\sqrt{h_b + h_c}} \leq \sqrt{\frac{\sum_{cyc}\sqrt{h_a + h_b}^2\sqrt{h_a + h_c}^2}{4F'^2}} = \sqrt{\frac{\sum_{cyc} h_b h_c}{4F'^2} + \frac{\left(\sum_{cyc} h_a\right)^2}{4F'^2}}$$

$$= \sqrt{1 + \frac{R}{2r}\left(\frac{h_a + h_b + h_c}{s}\right)^2},$$

which completes the proof. Equality holds iff ΔABC is equilateral.

1452. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}\ge\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a=y+z, b=z+x \text{ and } c=x+y \therefore 2s=a+b+c=2(x+y+z)$$

$$\Rightarrow s=x+y+z\therefore s-a=x, s-b=y\therefore\frac{r_a}{r_b}+\frac{r_b}{r_a}\ge\frac{a}{b}+\frac{b}{a}$$

$$\Leftrightarrow\frac{y}{x}+\frac{x}{y}\ge\frac{y+z}{z+x}+\frac{z+x}{y+z}\Leftrightarrow(x^2+y^2)(z+x)(y+z)\ge xy\left((z+x)^2+(y+z)^2\right)$$

$$\Leftrightarrow z(x^3+y^3)-xyz(x+y)+z^2(x-y)^2\ge 0\Leftrightarrow z(x+y)(x-y)^2+z^2(x-y)^2$$

$$\ge 0\to\text{true}\therefore\frac{r_a}{r_b}+\frac{r_b}{r_a}\ge\frac{a}{b}+\frac{b}{a}\text{ and analogs}\Rightarrow\frac{r_b}{r_c}+\frac{r_c}{r_b}\ge\frac{b}{c}+\frac{c}{b}$$

$$\therefore\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}\ge\sqrt{\frac{2m_a}{h_a}\left(\frac{b}{c}+\frac{c}{b}\right)}\overset{\text{Tereshin}}{\ge}\sqrt{\frac{2\left(\frac{b^2+c^2}{4R}\right)}{\frac{bc}{2R}}\left(\frac{b}{c}+\frac{c}{b}\right)}\Rightarrow\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}$$

$$\ge\frac{b}{c}+\frac{c}{b}\text{ and analogs}\therefore\sum_{cyc}\sqrt{\frac{2m_a}{h_a}\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)}\ge\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$$

$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1453. In any Δ ABC, the following relationship holds :

$$\frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\ge\frac{b}{c}+\frac{c}{b}\text{ and }\sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}\ge\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\ge\frac{b}{c}+\frac{c}{b}\Leftrightarrow\frac{s-b}{s-c}-1+\frac{s-c}{s-b}-1\ge\frac{b}{c}+\frac{c}{b}-2$$

$$\Leftrightarrow\frac{c-b}{s-c}+\frac{b-c}{s-b}\ge\frac{(b-c)^2}{bc}\Leftrightarrow(b-c)\left(\frac{1}{s-b}-\frac{1}{s-c}\right)\ge\frac{(b-c)^2}{bc}$$

$$\Leftrightarrow(b-c)\left(\frac{1}{s-b}-\frac{1}{s-c}\right)\ge\frac{(b-c)^2}{bc}\Leftrightarrow\frac{(b-c)^2}{(s-b)(s-c)}\ge\frac{(b-c)^2}{bc}$$

$$\Leftrightarrow\frac{(b-c)^2}{4bc(s-b)(s-c)}\left(4bc-4(s-b)(s-c)\right)\ge 0$$

$$\Leftrightarrow \frac{(b-c)^2}{4bc(s-b)(s-c)}\left(4bc-a^2+(b-c)^2\right)\geq 0$$

$$\Leftrightarrow \frac{(b-c)^2}{4bc(s-b)(s-c)}\left((b+c)^2-a^2\right)\geq 0 \Leftrightarrow \frac{(b-c)^2(b+c+a)(b+c-a)}{4bc(s-b)(s-c)}\geq 0$$

$$\to \text{true} \therefore \frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq \frac{b}{c}+\frac{c}{b}\ \forall\,\Delta\,ABC$$

$$\text{Let } \frac{a+c-b}{a+b-c}=x \text{ and } \frac{b}{c}=y \therefore \frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq \frac{b}{c}+\frac{c}{b}$$

$$\Rightarrow x+\frac{1}{x}\geq y+\frac{1}{y}\to (1)$$

$$\text{So, } \sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}\overset{?}{\geq}\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\Leftrightarrow \sqrt{x}+\frac{1}{\sqrt{x}}\overset{?}{\geq}\sqrt{y}+\frac{1}{\sqrt{y}}$$

$$\Leftrightarrow \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\overset{?}{\geq}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^2 \Leftrightarrow x+\frac{1}{x}+2\overset{?}{\geq}y+\frac{1}{y}+2\to \text{true via } (1)$$

$$\therefore \sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}\geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\ \forall\,\Delta\,ABC$$

$$\therefore \frac{a+b-c}{a+c-b}+\frac{a+c-b}{a+b-c}\geq \frac{b}{c}+\frac{c}{b} \text{ and } \sqrt{\frac{a+b-c}{a+c-b}}+\sqrt{\frac{a+c-b}{a+b-c}}$$

$$\geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1454. In ΔABC the following relationship holds:

$$2\left(\frac{2m_bm_c}{h_bh_c}-1\right)\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)\geq\left(\frac{b}{c}+\frac{c}{b}\right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Tereshin and CBS inequalities, we have

$$m_bm_c\geq\frac{(c^2+a^2)(a^2+b^2)}{(4R)^2}\geq\frac{(ca+ab)^2}{(4R)^2}=\frac{a^2(b+c)^2}{(4R)^2}.$$

and since $h_a=\dfrac{bc}{2R}$ (and analogs), then

$$\frac{2m_bm_c}{h_bh_c}-1\geq\frac{a^2(b+c)^2}{8R^2}\cdot\frac{4R^2}{a^2bc}-1=\frac{(b+c)^2}{2bc}-1=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right).$$

Now, we have

$$\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)-\left(\frac{b}{c}+\frac{c}{b}\right)=\left(\frac{s-c}{s-b}-\frac{c}{b}\right)-\left(\frac{b}{c}-\frac{s-b}{s-c}\right)=\frac{s(b-c)}{b(s-b)}-\frac{s(b-c)}{c(s-c)}=\frac{s(s-a)(b-c)^2}{bc(s-b)(s-c)}\geq 0,$$

$$\Rightarrow \frac{r_b}{r_c}+\frac{r_c}{r_b}\geq\frac{b}{c}+\frac{c}{b}.$$

Therefore

$$2\left(\frac{2m_bm_c}{h_bh_c}-1\right)\left(\frac{r_b}{r_c}+\frac{r_c}{r_b}\right)\ge\left(\frac{b}{c}+\frac{c}{b}\right)^2.$$

Equality holds iff ΔABC is equilateral.

1455. In ΔABC the following relationship holds:

$$\frac{r_a}{h_a^2}+\frac{r_b}{h_b^2}+\frac{r_c}{h_c^2}\ge\frac{4r}{R^2}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Daniel Sitaru-Romania

$$\frac{r_a}{h_a^2}+\frac{r_b}{h_b^2}+\frac{r_c}{h_c^2}=\frac{\frac{F}{s-a}}{\left(\frac{2F}{a}\right)^2}+\frac{\frac{F}{s-b}}{\left(\frac{2F}{b}\right)^2}+\frac{\frac{F}{s-c}}{\left(\frac{2F}{c}\right)^2}=$$

$$=\frac{1}{4F}\left(\frac{a^2}{s-a}+\frac{b^2}{s-b}+\frac{c^2}{s-c}\right)\overset{BERGSTROM}{\ge}$$

$$\ge\frac{1}{4F}\cdot\frac{(a+b+c)^2}{s-a+s-b+s-c}=\frac{1}{4F}\cdot\frac{4s^2}{s}=\frac{1}{4rs}\cdot\frac{4s^2}{s}=\frac{1}{r}=$$

$$=\frac{R^2}{R^2r}=\frac{R\cdot R}{R^2r}\overset{EULER}{\ge}\frac{2r\cdot 2r}{R^2r}=\frac{4r}{R^2}$$

Equality holds for $a=b=c$.

1456. In any Δ ABC, the following relationship holds :

$$\frac{6ar_a^2}{6+\sqrt{3}a(a+b+c)}+\frac{6br_b^2}{6+\sqrt{3}b(a+b+c)}+\frac{6cr_c^2}{6+\sqrt{3}c(a+b+c)}$$
$$\ge\frac{324Sr^2}{R(9+20S)-4Sr}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\ge b\ge c$ and then : $r_a^2\ge r_b^2\ge r_c^2$ and

$$\frac{6a}{6+\sqrt{3}a(a+b+c)} \geq \frac{6b}{6+\sqrt{3}b(a+b+c)} \geq \frac{6c}{6+\sqrt{3}c(a+b+c)}$$

$$\therefore \frac{6ar_a^2}{6+\sqrt{3}a(a+b+c)} + \frac{6br_b^2}{6+\sqrt{3}b(a+b+c)} + \frac{6cr_c^2}{6+\sqrt{3}c(a+b+c)}$$

$$\overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} r_a^2\right)\left(\sum_{cyc}\frac{6a}{6+\sqrt{3}a(a+b+c)}\right) \geq 2\left(\sum_{cyc} r_a r_b\right)\left(\sum_{cyc}\frac{1}{\frac{6}{a}+\sqrt{3}(2s)}\right)$$

$$\overset{\text{Bergstrom}}{\geq} 2s^2.\frac{9}{6\sum_{cyc}\frac{1}{a}+6\sqrt{3}s} = \frac{18s^2.4S}{\frac{6\sum_{cyc}ab}{R}+24\sqrt{3}sS} \overset{\text{Mitrinovic}}{\geq} \frac{72.27Sr^2}{\frac{6\sum_{cyc}a^2}{R}+24\sqrt{3}sS}$$

$$= \frac{324Sr^2}{\frac{\sum_{cyc}a^2}{R}+4\sqrt{3}sS} \overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\geq} \frac{324Sr^2}{9R+18RS} = \frac{324Sr^2}{9R+20RS-2RS} \overset{\text{Euler}}{\geq} \frac{324Sr^2}{9R+20RS-4rS}$$

$$\Rightarrow \frac{6ar_a^2}{6+\sqrt{3}a(a+b+c)} + \frac{6br_b^2}{6+\sqrt{3}b(a+b+c)} + \frac{6cr_c^2}{6+\sqrt{3}c(a+b+c)}$$

$$\geq \frac{324Sr^2}{R(9+20S)-4Sr}\ \forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1457. In any Δ ABC, the following relationship holds :

$$\frac{9r}{4Rp} \leq \sum_{cyc}\frac{\cos A}{b+c} \leq \frac{9}{8p}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

For own convenience, $p \equiv s$

$$\text{Now, } \cos B + \cos C = \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{bc^2+a^2b-b^3+ca^2+b^2c-c^3}{2abc}$$

$$= \frac{bc(b+c)-(b+c)(b^2-bc+c^2)+a^2(b+c)}{2abc}\ \therefore \sum_{cyc}\frac{\cos A}{b+c}$$

$$= \left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\frac{1}{b+c}\right) - \sum_{cyc}\frac{bc(b+c)-(b+c)(b^2-bc+c^2)+a^2(b+c)}{2abc(b+c)}$$

$$= \frac{R+r}{R}.\frac{\left(2\sum_{cyc}ab+\sum_{cyc}a^2\right)+\sum_{cyc}ab}{2s(s^2+2Rr+r^2)} - \frac{1}{8Rrs}\left(2\sum_{cyc}ab-\sum_{cyc}a^2\right)$$

$$= \frac{R+r}{R} \cdot \frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)} - \frac{2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)}{8Rrs}$$

$$\Rightarrow \sum_{cyc} \frac{\cos A}{b+c} \overset{(\bullet)}{=} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2+2Rr+r^2)}$$

$$\therefore \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9}{8s} \overset{\text{via } (\bullet)}{\Leftrightarrow} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2+2Rr+r^2)} \le \frac{9}{8s}$$

$$\Leftrightarrow (5R-16r)s^2 + 9R(2Rr+r^2) + 4Rr(4R+r) \overset{(*)}{\ge} 0$$

$\boxed{Case\ 1}$ $5R-16r \ge 0$ *and then* : LHS of $(*) \ge 9R(2Rr+r^2)+4Rr(4R+r) > 0$

$\Rightarrow (*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $5R-16r < 0$ *and then* : LHS of $(*)$

$$= -(16r-5R)s^2 + 9R(2Rr+r^2) + 4Rr(4R+r) \overset{\text{Gerretsen}}{\ge}$$

$$-(16r-5R)(4R^2+4Rr+3r^2) + 9R(2Rr+r^2) + 4Rr(4R+r) \overset{?}{\ge} 0$$

$$\Leftrightarrow 10t^3 - 5t^2 - 18t - 24 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(10t^2+15t+12) \overset{?}{\ge} 0 \rightarrow \text{true}$$

$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*)$ is true $\therefore$ combining *cases* 1 *and* 2, $(*)$ is true $\forall \Delta ABC$

$$\therefore \text{in any } \Delta ABC, \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9}{8p}$$

$$\text{Also}, \frac{9r}{4Rp} \le \sum_{cyc} \frac{\cos A}{b+c} \overset{\text{via } (\bullet)}{\Leftrightarrow} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2+2Rr+r^2)} \ge \frac{9r}{4Rs}$$

$$\Leftrightarrow (2R-r)s^2 - 9r(2Rr+r^2) - 2Rr(4R+r) \overset{(**)}{\ge} 0$$

Again, LHS of $(**) \overset{\text{Gerretsen}}{\ge} (2R-r)(16Rr-5r^2) - 9r(2Rr+r^2) - 2Rr(4R+r)$

$\overset{?}{\ge} 0 \Leftrightarrow 12R^2 - 23Rr - 2r^2 \overset{?}{\ge} 0 \Leftrightarrow (12R+r)(R-2r) \overset{?}{\ge} 0 \rightarrow$ true $\because R \overset{\text{Euler}}{\ge} 2r$

$\Rightarrow (**)$ is true $\forall \Delta ABC \therefore$ in *any* ΔABC, $\sum_{cyc} \frac{\cos A}{b+c} \ge \frac{9r}{4Rp} \therefore$ in *any* ΔABC,

$\frac{9r}{4Rp} \le \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9}{8p}$, *equalities* iff ΔABC is *equilateral* (QED)

1458. In acute triangle ABC holds:

$$\sum_{cyc} \tan B \tan C \ge 3 + \sum_{cyc} \sqrt{1+\tan^2 A}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x \coloneqq \tan A$, $y \coloneqq \tan B$, $z \coloneqq \tan C$. We have $x, y, z > 0$ and $x+y+z = xyz$.
By AM − GM inequality we have

$$3+\sum_{cyc}\sqrt{1+x^2}=3+\sum_{cyc}\sqrt{1+\frac{x(x+y+z)}{yz}}=$$

$$=3+\sum_{cyc}\sqrt{\frac{(x+y)(x+z)}{yz}}\le 3+\sum_{cyc}\frac{y+z}{x}=\sum_{cyc}x.\sum_{cyc}\frac{1}{x}=\sum_{cyc}yz,$$

as desired. Equality holds iff ΔABC is equilateral.

1459. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{(b^2+c^2)h_a}{b+c}\le\sum_{cyc}\frac{(b^2+c^2)r_a}{b+c}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\sum_{cyc}\frac{1}{a(b+c)}=\sum_{cyc}\frac{bc\left(\sum_{cyc}ab+a^2\right)}{4Rrs.2s(s^2+2Rr+r^2)}=\frac{\left(\sum_{cyc}ab\right)^2+8Rrs^2}{8Rrs^2(s^2+2Rr+r^2)}$$

$$\Rightarrow\sum_{cyc}\frac{1}{a(b+c)}\overset{(\bullet)}{=}\frac{\left(s^2+4Rr+r^2\right)^2+8Rrs^2}{8Rrs^2(s^2+2Rr+r^2)}$$

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a-2s+2s}{2s-a}=-3+\frac{2s}{2s(s^2+2Rr+r^2)}.\sum_{cyc}\left(\sum_{cyc}ab+a^2\right)$$

$$=-3+\frac{2s}{2s(s^2+2Rr+r^2)}.\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)=-3+\frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}$$

$$\Rightarrow\sum_{cyc}\frac{a}{b+c}\overset{(\bullet\bullet)}{=}\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}$$

$$\sum_{cyc}\mathrm{cosec}^2\frac{A}{2}=3+\sum_{cyc}\cot^2\frac{A}{2}=3+\sum_{cyc}\frac{s^2}{r_a^2}$$

$$=3+\frac{s^2}{r^2s^4}\left(\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\left(\sum_{cyc}r_a\right)\right)=3+\frac{s^2\left(s^4-2rs^2(4R+r)\right)}{r^2s^4}$$

$$\Rightarrow \sum_{cyc} \text{cosec}^2 \frac{A}{2} \overset{(\bullet\bullet\bullet)}{=} \frac{s^2 - 8Rr + r^2}{r^2}$$

$$\sum_{cyc} b^2c^2r_a \overset{\text{via (i)}}{=} \sum_{cyc} b^2c^2 \left(4R + r - 4R\cos^2 \frac{A}{2}\right)$$

$$= (4R + r)\sum_{cyc} b^2c^2 - \sum_{cyc} \frac{16R^2r^2s^2 . 4R\cos^2 \frac{A}{2}}{16R^2 \cos^2 \frac{A}{2} \sin^2 \frac{A}{2}}$$

$$= (4R + r)\sum_{cyc} b^2c^2 - 4Rr^2s^2 \sum_{cyc} \text{cosec}^2 \frac{A}{2}$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{=} (4R + r)\sum_{cyc} a^2b^2 - 4Rr^2s^2 . \frac{s^2 - 8Rr + r^2}{r^2}$$

$$\Rightarrow \sum_{cyc} b^2c^2r_a \overset{(\bullet\bullet\bullet\bullet)}{=} (4R + r)\sum_{cyc} a^2b^2 - 4Rs^2(s^2 - 8Rr + r^2)$$

$$\sum_{cyc} a^2r_a = rs\sum_{cyc} \frac{(a - s + s)^2}{s - a} = rs\left(\sum_{cyc}(s - a) - 6s + \frac{s^2}{r^2s}\sum_{cyc}(s - b)(s - c)\right)$$

$$= rs\left(-5s + \frac{s^2(4Rr + r^2)}{r^2s}\right) \Rightarrow \sum_{cyc} a^2r_a \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} (4R - 4r)s^2$$

$$\therefore \sum_{cyc} \frac{(b^2 + c^2)h_a}{b + c} = \sum_{cyc} \frac{(\sum_{cyc} a^2 - a^2)h_a}{b + c}$$

$$= 2rs\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} \frac{1}{a(b + c)}\right) - 2rs\sum_{cyc} \frac{a}{b + c}$$

$$\overset{\text{via } (\bullet),(\bullet\bullet)}{=} \frac{2rs\left((s^2 - 4Rr - r^2)\left((s^2 + 4Rr + r^2)^2 + 8Rrs^2\right) - 4Rrs^2(2s^2 - 2Rr - 2r^2)\right)}{4Rrs^2(s^2 + 2Rr + r^2)}$$

$$\Rightarrow \boxed{\sum_{cyc} \frac{(b^2 + c^2)h_a}{b + c} \overset{(1)}{=} \frac{s^6 + (4Rr + r^2)s^4 - r^2s^2(40R^2 + 8Rr + r^2) - (4Rr + r^2)^3}{2Rs(s^2 + 2Rr + r^2)}} \text{ and}$$

$$\sum_{cyc} \frac{(b^2 + c^2)r_a}{b + c} = \sum_{cyc} \frac{(a^2 + \sum_{cyc} ab)(b^2 + c^2)r_a}{2s(s^2 + 2Rr + r^2)}$$

$$= \frac{1}{2s(s^2 + 2Rr + r^2)} \left(\begin{array}{c} \sum_{cyc}\left(\left(\sum_{cyc} a^2b^2 - b^2c^2\right)r_a\right) + \\ \left(\sum_{cyc} ab\right)\sum_{cyc}\left(\left(\sum_{cyc} a^2 - a^2\right)r_a\right) \end{array} \right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \left(\begin{array}{c} \left(\sum_{cyc} a^2b^2\right)(4R+r) - \sum_{cyc} b^2c^2r_a + \\ \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)(4R+r) - \left(\sum_{cyc} ab\right)\sum_{cyc} a^2r_a \end{array} \right)$$

$$\overset{via\ (\bullet\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet\bullet)}{=} \frac{1}{2s(s^2+2Rr+r^2)} \left(\begin{array}{c} \left(\sum_{cyc} a^2b^2\right)(4R+r) - (4R+r)\sum_{cyc} a^2b^2 + \\ 4Rs^2(s^2-8Rr+r^2) + \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)(4R+r) - \\ \left(\sum_{cyc} ab\right)(4R-4r)s^2 \end{array} \right)$$

$$\Rightarrow \boxed{\sum_{cyc} \frac{(b^2+c^2)r_a}{b+c} \overset{(2)}{=} \frac{(4R+3r)s^4 - rs^2(24R^2-8Rr-2r^2) - r^2(4R+r)^3}{s(s^2+2Rr+r^2)}}$$

$$\therefore (1),(2) \Rightarrow \sum_{cyc} \frac{(b^2+c^2)h_a}{b+c} \le \sum_{cyc} \frac{(b^2+c^2)r_a}{b+c}$$

$$\Leftrightarrow \frac{s^6 + (4Rr+r^2)s^4 - r^2s^2(40R^2+8Rr+r^2) - (4Rr+r^2)^3}{2Rs(s^2+2Rr+r^2)}$$

$$\le \frac{(4R+3r)s^4 - rs^2(24R^2-8Rr-2r^2) - r^2(4R+r)^3}{s(s^2+2Rr+r^2)}$$

$$\Leftrightarrow \boxed{\begin{array}{c} s^6 - (8R^2+2Rr-r^2)s^4 + rs^2(48R^3-56R^2r-12Rr^2-r^3) + \\ r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4) \overset{(*)}{\le} 0 \end{array}}$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\le} (4R^2+4Rr+3r^2)s^4 - (8R^2+2Rr-r^2)s^4$$
$$+rs^2(48R^3-56R^2r-12Rr^2-r^3)$$
$$+r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4) \overset{?}{\le} 0$$
$$\Leftrightarrow (4R^2-2Rr-4r^2)s^4 - rs^2(48R^3-56R^2r-12Rr^2-r^3)$$
$$\underset{(**)}{\underbrace{\overset{?}{\ge}}} r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4)$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\ge} (4R^2-2Rr-4r^2)(16Rr-5r^2)s^2$$
$$-rs^2(48R^3-56R^2r-12Rr^2-r^3) = r(16R^3+4R^2r-42Rr^2+21r^3)s^2$$
$$\overset{?}{\ge} r^2(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4) \Leftrightarrow$$
$$(16R^3+4R^2r-42Rr^2+21r^3)s^2 \underset{(***)}{\underbrace{\overset{?}{\ge}}} r(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4)$$

$$\text{Once again, LHS of } (***) \overset{\text{Gerretsen}}{\ge}$$

$$(16R^3 + 4R^2r - 42Rr^2 + 21r^3)(16Rr - 5r^2) \overset{?}{\geq}$$
$$r(128R^4 + 32R^3r - 24R^2r^2 - 10Rr^3 - r^4)$$
$$\Leftrightarrow 32t^4 - 12t^3 - 167t^2 + 139t - 26 \overset{?}{\geq} 0 \ \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(32t^3 + 20t^2 + 32t(t-2) + t + 13) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \sum_{\text{cyc}} \frac{(b^2+c^2)h_a}{b+c} \leq \sum_{\text{cyc}} \frac{(b^2+c^2)r_a}{b+c} \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1460. In any acute triangle ABC with semiperimeter p holds

$$a\sqrt{\cos A} + b\sqrt{\cos B} + c\sqrt{\cos C} \leq p\sqrt{2}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$a\sqrt{\cos A} + b\sqrt{\cos B} + c\sqrt{\cos C} \leq \sqrt{(a+b+c)(a\cos A + b\cos B + c\cos C)},$$

and since $a+b+c = 2p$ and $a\cos A + b\cos B + c\cos C = \frac{2F}{R} = p.\frac{2r}{R} \overset{Euler}{\leq} p$, then

$$a\sqrt{\cos A} + b\sqrt{\cos B} + c\sqrt{\cos C} \leq p\sqrt{2},$$

Equality holds iff ΔABC is equilateral.

1461. In any acute Δ ABC, the following relationship holds :

$$\frac{a}{h_a}\sqrt{\tan A} + \frac{b}{h_b}\sqrt{\tan B} + \frac{c}{h_c}\sqrt{\tan C} > 2\sqrt[4]{27}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $f(x) = (\sin x)\sqrt{\tan x} \ \forall x \in \left(0, \frac{\pi}{2}\right)$ and then : $f''(x)$

$$= \frac{(4\sec^2 x - 4)(\sin x)(\tan^2 x) + 4(\sin x)(\sec^2 x) - (\sec^4 x)(\sin x)}{4\tan^{\frac{3}{2}} x}$$
$$= \frac{4(\sin x)(\tan^4 x) + 3(\sin x)(\sec^2 x) - (\sin x)(\sec^2 x)(\sec^2 x - 1)}{4\tan^{\frac{3}{2}} x}$$
$$= \frac{4(\sin x)(\tan^4 x) + 3\sin x + 3(\sin x)(\tan^2 x) - (\sin x)(\sec^2 x)(\tan^2 x)}{4\tan^{\frac{3}{2}} x}$$

$$= \frac{4(\sin x)(\tan^4 x) + 3\sin x + 3(\sin x)(\tan^2 x) - (\sin x)(\tan^2 x) - (\sin x)(\tan^4 x)}{4\tan^{\frac{3}{2}} x}$$

$$= \frac{3(\sin x)(\tan^4 x) + 3\sin x + 2(\sin x)(\tan^2 x)}{4\tan^{\frac{3}{2}} x} > 0 \Rightarrow f(x) \text{ is convex} \to (1)$$

Now, WLOG if we assume $a \geq b \geq c$, then : $a\sqrt{\tan A} \geq b\sqrt{\tan B} \geq c\sqrt{\tan C}$

$(\because \Delta ABC$ is acute$)$ and $\frac{1}{h_a} \geq \frac{1}{h_b} \geq \frac{1}{h_c}$ ∴ via Chebyshev,

$$\frac{a}{h_a}\sqrt{\tan A} + \frac{b}{h_b}\sqrt{\tan B} + \frac{c}{h_c}\sqrt{\tan C} \geq \frac{1}{3}\left(\sum_{cyc}\frac{1}{h_a}\right)\left(\sum_{cyc} a\sqrt{\tan A}\right) \overset{\text{Jensen,via }(1)}{\geq}$$

$$\frac{3.2R}{3r}.\sin\frac{\pi}{3}.\sqrt{\tan\frac{\pi}{3}} = \frac{2R}{r}.\frac{\sqrt{3}}{2}.\sqrt{\sqrt{3}} \overset{\text{Euler}}{\geq} 2.3^{\frac{3}{4}} = 2\sqrt[4]{27}$$

$\forall$ acute ΔABC, $'' = ''$ iff ΔABC is equilateral (QED)

1462. In any ΔABC, the following relationship holds :

$$\frac{(h_a^3 + h_a^2 w_b + w_b^3)^5}{(w_a^3 + w_a^2 m_b + m_b^3)^3} + \frac{(h_b^3 + h_b^2 w_c + w_c^3)^5}{(w_b^3 + w_b^2 m_c + m_c^3)^3} + \frac{(h_c^3 + h_c^2 w_a + w_a^3)^5}{(w_c^3 + w_c^2 m_a + m_a^3)^3} \geq \frac{6^9 \cdot r^{15}}{(9R^3 - 64r^3)^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(h_a^3 + h_a^2 w_b + w_b^3)^5}{(w_a^3 + w_a^2 m_b + m_b^3)^3} + \frac{(h_b^3 + h_b^2 w_c + w_c^3)^5}{(w_b^3 + w_b^2 m_c + m_c^3)^3} + \frac{(h_c^3 + h_c^2 w_a + w_a^3)^5}{(w_c^3 + w_c^2 m_a + m_a^3)^3}$$

$$\geq \sum_{cyc}\frac{(h_a^3 + h_a^2 h_b + h_b^3)^5}{(m_a^3 + m_a^2 m_b + m_b^3)^3} \overset{\text{Panaitopol}}{\geq} \sum_{cyc}\frac{(h_a^3 + h_a^2 h_b + h_b^3)^5}{\left(\frac{R^9}{512r^9} * (h_a^3 + h_a^2 h_b + h_b^3)\right)^3}$$

$$\geq \frac{512r^9}{R^9}\sum_{cyc}\left(h_a h_b(h_a + h_b) + h_a^2 h_b\right)^2 = \frac{512r^9}{R^9} h_a^2 h_b^2 h_c^2 \sum_{cyc}\left(\frac{2h_a + h_b}{h_c}\right)^2 \geq$$

$$\frac{512r^9}{R^9} * \frac{4r^4 s^4}{R^2} * \frac{1}{3}\left(2\sum_{cyc}\frac{h_b}{h_a} + \sum_{cyc}\frac{h_a}{h_b}\right)^2 \overset{\text{Gerretsen}}{\geq}$$

$$\frac{512r^9}{R^9} * r^4\left(\frac{27Rr + 5r(R - 2r)}{R}\right)^2 * \frac{1}{3}\left(2\sum_{cyc}\frac{h_b}{h_a} + \sum_{cyc}\frac{h_a}{h_b}\right)^2 \overset{\substack{\text{Euler}\\\text{and}\\\text{A-G}}}{\geq}$$

$$\frac{512r^9}{R^9} * r^4 * 729r^2 * \frac{1}{3} * (6+3)^2 = \frac{6^9 * r^{15}}{R^9} \overset{?}{\geq} \frac{6^9 * r^{15}}{(9R^3 - 64r^3)^3} \Leftrightarrow 9R^3 - 64r^3 \overset{?}{\geq} R^3$$

$$\Leftrightarrow 8R^3 \overset{?}{\geq} 64r^3 \Leftrightarrow R \overset{?}{\geq} 2r \to \text{true via Euler} \therefore \frac{(h_a^3 + h_a^2 w_b + w_b^3)^5}{(w_a^3 + w_a^2 m_b + m_b^3)^3} +$$

$$\frac{(h_b^3+h_b^2w_c+w_c^3)^5}{(w_b^3+w_b^2m_c+m_c^3)^3}+\frac{(h_c^3+h_c^2w_a+w_a^3)^5}{(w_c^3+w_c^2m_a+m_a^3)^3}\ge\frac{6^9*r^{15}}{(9R^3-64r^3)^3}$$

$\forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC$ is equilateral (QED)

1463. In any $\Delta\,ABC$, the following relationship holds:

$$\frac{r_a^5}{h_a^2w_a^2m_a^2}+\frac{r_b^5}{h_b^2w_b^2m_b^2}+\frac{r_c^5}{h_c^2w_c^2m_c^2}\ge\frac{64\cdot r^5}{(9R^3-64r^3)^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\ge b\ge c$ and then: $r_a\ge r_b\ge r_c;\ m_a\le m_b\le m_c;$

$$\frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c}\Rightarrow\frac{r_a^5}{m_a^4}\ge\frac{r_b^5}{m_b^4}\ge\frac{r_c^5}{m_c^4}\text{ and }\frac{1}{h_a^2}\ge\frac{1}{h_b^2}\ge\frac{1}{h_c^2}\to(1)$$

$$\text{Now, }\frac{r_a^5}{h_a^2w_a^2m_a^2}+\frac{r_b^5}{h_b^2w_b^2m_b^2}+\frac{r_c^5}{h_c^2w_c^2m_c^2}\ge\sum_{cyc}\left(\frac{1}{h_a^2}*\frac{r_a^5}{m_a^4}\right)\overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}\left(\sum_{cyc}\frac{1}{h_a^2}\right)\left(\sum_{cyc}\frac{r_a^5}{m_a^4}\right)\overset{\text{Radon}}{\ge}\frac{1}{3*4r^2s^2}\left(\sum_{cyc}a^2\right)\frac{\left(\sum_{cyc}r_a\right)^5}{\left(\sum_{cyc}m_a\right)^4}\overset{\substack{\text{Leuenberger}\\ \text{and}\\ \text{Ionescu-Weitzenbock + Mitrinovic}}}{\ge}$$

$$\frac{1}{3*4r^2s^2}*\left(4\sqrt{3}r*3\sqrt{3}r\right)*\frac{\left(\sum_{cyc}r_a\right)^5}{\left(\sum_{cyc}r_a\right)^4}=\frac{3(4R+r)}{s^2}$$

$$\overset{\text{Mitrinovic}}{\ge}\frac{3(4R+r)}{\frac{27R^2}{4}}\overset{?}{\ge}\frac{64*r^5}{(9R^3-64r^3)^2}$$

$$\Leftrightarrow 324t^7+81t^6-4608t^4-1152t^3-144t^2+16384t+4096\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)\begin{pmatrix}324t^5+1377t^4+4212t^3\\+6732t^2+8928t+8640\end{pmatrix}+15232\right)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\therefore\frac{r_a^5}{h_a^2w_a^2m_a^2}+\frac{r_b^5}{h_b^2w_b^2m_b^2}+\frac{r_c^5}{h_c^2w_c^2m_c^2}\ge\frac{64*r^5}{(9R^3-64r^3)^2}$$

$\forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC$ is equilateral (QED)

1464. In any $\Delta\,ABC$, the following relationship holds:

$$\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}\le\frac{27}{32}\cdot\frac{81R^5-2560r^5}{r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}\le\frac{m_a^3}{h_b}+\frac{m_b^3}{h_c}+\frac{m_c^3}{h_a}\overset{\text{Panaitopol}}{\le}$$

$$\left(\frac{\frac{Rs}{a}}{\frac{2rs}{b}}\right)\left(\frac{2b^2+2c^2+2a^2-3a^2}{4}\right)+\left(\frac{\frac{Rs}{b}}{\frac{2rs}{c}}\right)\left(\frac{2c^2+2a^2+2b^2-3b^2}{4}\right)$$

$$+\left(\frac{\frac{Rs}{c}}{\frac{2rs}{a}}\right)\left(\frac{2a^2+2b^2+2c^2-3c^2}{4}\right)=\frac{R}{2r}\left(\left(\frac{\sum_{cyc}a^2}{2}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\frac{3}{4}\sum_{cyc}ab\right)$$

$$\overset{\text{Leibnitz}}{\le}\frac{R}{2r}\cdot\frac{9R^2}{2}\cdot\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\frac{3R}{8r}\cdot 4Rrs\cdot\sum_{cyc}\frac{1}{a}\overset{\substack{\text{CBS}\\ \text{and}\\ \text{Bergstrom}}}{\le}\frac{R}{2r}\cdot\frac{9R^2}{2}\cdot\sqrt{\sum_{cyc}a^2\cdot\frac{\sum_{cyc}a^2b^2}{16R^2r^2s^2}}$$

$$-\frac{3R}{8r}\cdot 4Rrs\cdot\frac{9}{2s}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Goldstone}}}{\le}\frac{R}{2r}\cdot\frac{9R^2}{2}\cdot\sqrt{9R^2\cdot\frac{4R^2s^2}{16R^2r^2s^2}}-\frac{3R}{8r}\cdot 4Rrs\cdot\frac{9}{2s}=\frac{27R^4}{8r^2}-\frac{27R^2}{4}$$

$$\overset{?}{\le}\frac{27}{32}\cdot\frac{81R^5-2560r^5}{r^3}\Leftrightarrow 81R^5-4R^4r+8R^2r^3-2560r^5\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)\left(81R^4+158R^3r+316R^2r^2+640Rr^3+1280r^4\right)\overset{?}{\ge}0\to\text{true}$$

$$\because R\overset{\text{Euler}}{\ge}2r\therefore\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}\le\frac{27}{32}\cdot\frac{81R^5-2560r^5}{r^3}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since

$h_a\le w_a\le m_a$ (and analogs), and by using Rearrangement inequality, we have

$$\frac{m_a^3}{w_b}+\frac{w_b^3}{h_c}+\frac{h_c^3}{m_a}=\frac{h_cm_a^4+m_aw_b^4+w_bh_c^4}{m_aw_bh_c}\le\frac{m_cm_a^4+m_am_b^4+m_bm_c^4}{h_ah_bh_c}$$

$$\le\frac{m_a^5+m_b^5+m_c^5}{h_ah_bh_c}.$$

Also, we have

$$\bullet\ h_ah_bh_c=\frac{2s^2r^2}{R}\overset{\text{Cosnita-Turtoiu}}{\ge}27r^3\ \ (1)$$

$$\bullet\ m_a^5+m_b^5+m_c^5=\left(\sum_{cyc}m_a\right)^5-5\left(\sum_{cyc}m_a^2+\sum_{cyc}m_bm_c\right)\prod_{cyc}(m_b+m_c)$$

$$\overset{\text{Leuenberger \& AM-GM}}{\le}(4R+r)^5-5.6\sqrt[3]{(m_am_bm_c)^2}.8m_am_bm_c$$

$$\overset{Euler\ \&\ m_a \ge h_a}{\leq} \left(\frac{9R}{2}\right)^5 - 240\sqrt[3]{(h_a h_b h_c)^5} \overset{(1)}{\leq} \left(\frac{9R}{2}\right)^5 - 240(3r)^5$$

$$= \frac{27^2(81R^5 - 2560r^5)}{32}.$$

Using these results, we obtain

$$\frac{{m_a}^3}{w_b} + \frac{{w_b}^3}{h_c} + \frac{{h_c}^3}{m_a} < \frac{27}{32} \cdot \frac{81R^5 - 2560r^5}{r^3}.$$

Equality holds iff ΔABC is equilateral.

1465. In ΔABC the following relationship holds:

$$\frac{(a^4+b^4)h_c}{a+b} + \frac{(b^4+c^4)h_a}{b+c} + \frac{(c^4+a^4)h_b}{c+a} \ge 8\sqrt{3} \cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase – Romania

Solution 1 by Tapas Das-India

$$\frac{(a^4+b^4)h_c}{a+b} + \frac{(b^4+c^4)h_a}{b+c} + \frac{(c^4+a^4)h_b}{(c+a)}$$

$$\overset{AM-GM}{\geq} \frac{2a^2b^2 \cdot 2F}{ac+bc} + \frac{2b^2c^2 \cdot 2F}{ab+ac} + \frac{2c^2a^2 \cdot 2F}{bc+ab}$$

$$= 4F\left[\frac{a^2b^2}{ac+bc} + \frac{b^2c^2}{ab+ac} + \frac{c^2a^2}{bc+ab}\right] \ge 4F\frac{(ab+bc+ca)^2}{2(ab+bc+ca)}$$

$$= 2F\,(ab+bc+ca) \ge 2F\ \times 4\sqrt{3}F = 8\sqrt{3}F^2$$

Note: $ab+bc+ca \ge 4\sqrt{3}F$

$$ab+bc+ca \ge 3[(abc)^2]^{\frac{1}{3}} \ge 3\left(\frac{4F}{\sqrt{3}}\right) = 4\sqrt{3}F$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{(a^2)^2 + (b^2)^2}{\frac{a+b}{h_c}} \overset{CBS}{\geq} \frac{(2\sum a^2)^2}{2\sum\frac{a+b}{h_c}} =$$

$$= \frac{4(\sum a^2)^2}{\frac{2}{2F}\cdot\sum(a+b)c} = \frac{2F(\sum a^2)^2}{\sum ab} \ge \frac{2F(\sum a^2)^2}{\sum a^2} \ge 2F \cdot 4\sqrt{3}F = 8\sqrt{3}F^2$$

1466. In ΔABC the following relationship holds:

$$\left(\frac{m_a}{w_b}+\frac{w_b}{h_c}\right)^2+\left(\frac{m_b}{w_c}+\frac{w_c}{h_a}\right)^2+\left(\frac{m_c}{w_a}+\frac{w_a}{h_b}\right)^2\ge\frac{192r^4}{3R^4-32r^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\left(\frac{m_a}{w_b}+\frac{w_b}{h_c}\right)^2+\left(\frac{m_b}{w_c}+\frac{w_c}{h_a}\right)^2+\left(\frac{m_c}{w_a}+\frac{w_a}{h_b}\right)^2\overset{AM-GM}{\ge}$$

$$\ge\left(2\sqrt{\frac{m_a}{w_b}\cdot\frac{w_b}{h_c}}\right)^2+\left(2\sqrt{\frac{m_b}{w_c}\cdot\frac{w_c}{h_a}}\right)^2+\left(2\sqrt{\frac{m_c}{w_a}\cdot\frac{w_a}{h_b}}\right)^2=$$

$$=4\sum_{cyc}\frac{m_a}{h_c}=4\sum_{cyc}\frac{m_a}{\frac{2F}{c}}=\frac{2}{F}\sum_{cyc}cm_a\overset{CEBYSHEV}{\ge}\ge\frac{2}{rs}\cdot\frac{1}{3}\sum_{cyc}a\sum_{cyc}m_a\overset{GOTMAN}{\ge}\frac{2}{rs}\cdot\frac{2s}{3}\cdot 9r$$

$$=12$$

Remains to prove:

$$12\ge\frac{192r^4}{3R^4-32r^4},\qquad 12(3R^4-32r^4)\ge 192r^4$$

$$3R^4-32r^4\ge 16r^4\Leftrightarrow 3R^4\ge 48r^4\Leftrightarrow R^4\ge 16r^4\Leftrightarrow R\ge 2r\ (Euler)$$

Equality holds for: $a=b=c$.

1467.

In any Δ ABC and $\forall$ m, n $\in\mathbb{N}$; n $\ge$ 2, the following relationship holds :

$$\frac{(h_a^m+w_b^m+m_c^m)^n}{r_a}+\frac{(h_b^m+w_c^m+m_a^m)^n}{r_b}+\frac{(h_c^m+w_a^m+m_b^m)^n}{r_c}\ge\frac{2.3^{n(m+1)}.r^{mn}}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(h_a^m+w_b^m+m_c^m)^n}{r_a}+\frac{(h_b^m+w_c^m+m_a^m)^n}{r_b}+\frac{(h_c^m+w_a^m+m_b^m)^n}{r_c}\overset{Holder}{\ge}$$

$$\frac{\left(\sum_{cyc}h_a^m+\sum_{cyc}w_a^m+\sum_{cyc}m_a^m\right)^n}{3^{n-2}\left(\sum_{cyc}r_a\right)}\ge\frac{\left(\sum_{cyc}h_a^m+\sum_{cyc}h_a^m+\sum_{cyc}h_a^m\right)^n}{3^{n-2}(4R+r)}\overset{\substack{Holder\\and\\Euler}}{\ge}$$

$$\frac{3^n\left(\frac{1}{3^{m-1}}\left(\sum_{cyc} h_a\right)^m\right)^n}{3^{n-2}\cdot\frac{9R}{2}} = \frac{2}{3^{mn-n}\cdot R}\left(\sum_{cyc}\frac{2rs}{a}\right)^{mn} \overset{Bergstrom}{\geq} \frac{2}{3^{mn-n}\cdot R}\left(\frac{2rs\cdot 9}{2s}\right)^{mn}$$

$$= \frac{2r^{mn}\cdot 3^{2mn}}{3^{mn-n}\cdot R} = \frac{2\cdot 3^{n(m+1)}\cdot r^{mn}}{R}\ \forall\Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1468. In any Δ ABC, the following relationship holds :

$$\frac{m_a w_b}{w_c h_a} + \frac{w_b h_c}{h_a m_b} + \frac{h_c m_a}{m_b w_c} \leq \frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3 - 64\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a w_b}{w_c h_a} + \frac{w_b h_c}{h_a m_b} + \frac{h_c m_a}{m_b w_c} \leq \frac{m_a m_b}{h_c h_a} + \frac{m_b m_c}{h_a h_b} + \frac{m_c m_a}{h_b h_c}$$

$$\leq m_a m_b m_c\left(\frac{1}{h_c^2 h_a} + \frac{1}{h_a^2 h_b} + \frac{1}{h_b^2 h_c}\right) \leq \frac{Rs^2}{2}\cdot\frac{c^2a + a^2b + b^2c}{8r^3s^3}$$

$$\overset{A-G}{\leq} \frac{R}{16r^3s}\cdot(a^3+b^3+c^3) = \frac{2sR(s^2-6Rr-3r^2)}{16r^3s}$$

$$\overset{Gerretsen}{\leq} \frac{R(4R^2-2Rr)}{8r^3} \overset{?}{\leq} \frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3 - 64\right) \Leftrightarrow 23R^3 + 2R^2r - 192r^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(23R^2+48Rr+96r^2) \overset{?}{\geq} 0 \to \text{true} \because R \overset{Euler}{\geq} 2r$$

$$\therefore \frac{m_a w_b}{w_c h_a} + \frac{w_b h_c}{h_a m_b} + \frac{h_c m_a}{m_b w_c} \leq \frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3 - 64\right)\ \forall\Delta ABC,$$

$$"=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Proof of $m_a m_b m_c \leq \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\overset{(3)}{=}$$

$$\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\left(\begin{matrix}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{matrix}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\leq\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\leq}0$$

Now, LHS of $(\bullet)\overset{\text{Gerretsen}}{\leq}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$

$-r^3(4R+r)^3\overset{?}{\leq}0$

$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\underset{(\bullet\bullet)}{\overset{?}{\geq}}20rs^4$

Now, LHS of $(\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)}s^2(16Rr-5r^2)(8R-16r)$

$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$ *and*

$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2(4R^2+4Rr+3r^2)$

$(a),(b) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$

$\geq 20rs^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$

$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$

Now, LHS of $(\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)} (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2(4R^2+4Rr+3r^2)$

$(c),(d) \Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

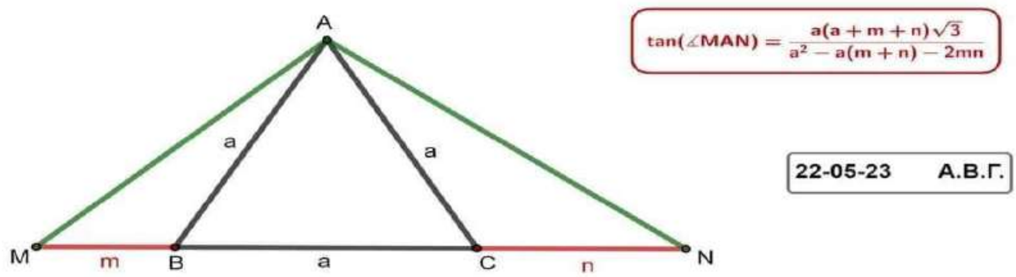
$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \geq 27r^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow 224t^3-587t^2+308t-60 \geq 0 \left(\text{where } t=\frac{R}{r}\right)$

$\Leftrightarrow (t-2)\left((t-2)(224t+309)+648\right) \geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$

$\Rightarrow (\bullet)$ is true $\Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2}$ (QED)

1469.

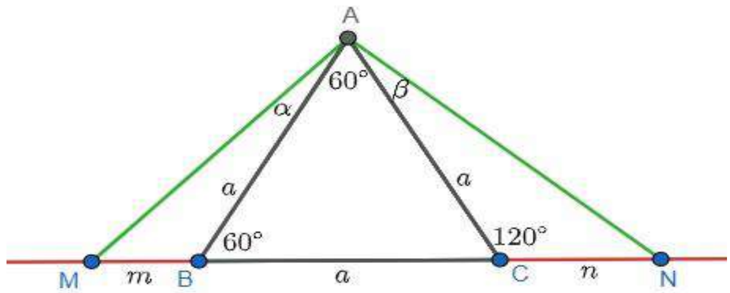


With reference to diagram, prove that :

$$\tan(\sphericalangle MAN)=\frac{\sqrt{3}.4a(a+m+n)}{4(a^2-a(m+n)-2mn)}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Via cotangent rule}: (m+a)\cot 60^\circ = m\cot\alpha - a\cot 60^\circ$$

$$\Rightarrow \frac{a+m}{\sqrt{3}} = \frac{m}{\tan\alpha} - \frac{a}{\sqrt{3}} \Rightarrow \tan\alpha \overset{(*)}{=} \frac{\sqrt{3}m}{2a+m}$$

$$\text{Also, via cotangent rule}: (a+n)\cot 120^\circ = a\cot 60^\circ - n\cot\beta$$

$$\Rightarrow \frac{a+n}{-\sqrt{3}} = \frac{a}{\sqrt{3}} - \frac{n}{\tan\beta} \Rightarrow \tan\beta \overset{(**)}{=} \frac{\sqrt{3}n}{2a+n}$$

$$\therefore \tan(\measuredangle MAN) = \frac{\tan\alpha+\tan\beta+\tan 60^\circ-\tan\alpha.\tan\beta.\tan 60^\circ}{1-\tan\alpha.\tan\beta-\tan\alpha.\tan 60^\circ-\tan\beta.\tan 60^\circ} \overset{\text{via }(*),(**)}{=}$$

$$\frac{\frac{\sqrt{3}m}{2a+m}+\frac{\sqrt{3}n}{2a+n}+\sqrt{3}-\frac{\sqrt{3}m}{2a+m}.\frac{\sqrt{3}n}{2a+n}.\sqrt{3}}{1-\frac{\sqrt{3}m}{2a+m}.\frac{\sqrt{3}n}{2a+n}-\frac{\sqrt{3}m}{2a+m}.\sqrt{3}-\frac{\sqrt{3}n}{2a+n}.\sqrt{3}} = \frac{\sqrt{3}.4a(a+m+n)}{4(a^2-am-an-2mn)}$$

$$\Rightarrow \tan(\measuredangle MAN) = \frac{\sqrt{3}.4a(a+m+n)}{4(a^2-a(m+n)-2mn)}\ (\text{QED})$$

1470. In any Δ ABC, the following relationship holds :

$$\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3 \ge \frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3 \overset{A-G}{\ge}$$

$$3\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right) \overset{A-G}{\ge} 3.8.\sqrt{\frac{m_a}{w_b}.\frac{w_c}{h_a}.\frac{m_b}{w_c}.\frac{w_a}{h_b}.\frac{m_c}{w_a}.\frac{w_b}{h_c}} \ge 3.8$$

$$\overset{?}{\ge} \frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6} \Leftrightarrow (9R^3-64r^3)^2 \overset{?}{\ge} 64r^6 \Leftrightarrow 9R^3-64r^3 \overset{?}{\ge} 8r^3$$

$$\Leftrightarrow R^3 \overset{?}{\geq} 8r^3 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3$$
$$\geq \frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6} \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1471.

In any bicentric quadrilateral ABCD with sides a,b,c,d, the following relationship holds:

$$3R\sqrt{2}.\min\left(\frac{1}{a^{-3}+b^{-3}+c^{-3}},\frac{1}{a^{-3}+b^{-3}+d^{-3}},\frac{1}{a^{-3}+c^{-3}+d^{-3}},\frac{1}{b^{-3}+c^{-3}+d^{-3}}\right)$$
$$\leq F^2 \leq \frac{R\sqrt{2}}{3}.\max(a^3+b^3+c^3,a^3+b^3+d^3,a^3+c^3+d^3,b^3+c^3+d^3)$$

Proposed by Emil C. Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

Via Brahmagupta and Parameshvara,

$$16F^2R^2=(ac+bd)(ab+cd)(ad+bc)$$
$$\Rightarrow x\left(bd\left((a+c)^2-2ac\right)+ac\left((b+d)^2-2bd\right)\right)=16R^2r^2s^2 \; (x=ac+bd)$$
$$\Rightarrow x\left(bd(s^2-2ac)+ac(s^2-2bd)\right)=16R^2r^2s^2 \Rightarrow x(s^2x-4r^2s^2)=16R^2r^2s^2$$
$$\Rightarrow x^2-4x.r^2-16R^2r^2=0 \Rightarrow x=\frac{4r^2\pm\sqrt{64R^2r^2+16r^4}}{2}$$
$$\Rightarrow ac+bd=2r^2+2r*\sqrt{4R^2+r^2} \to (1)$$

Now, $(bs+ca)(as+bd)(ds+ca)(cs+bd)$

$$=s^3(a^2bc^2+a^2c^2d+ab^2d^2+b^2cd^2)$$
$$+s^2(a^3c^3+b^3d^3+a^2b^2cd+a^2bcd^2+ab^2c^2d+abc^2d^2)$$
$$+sabcd(a^2c+ac^2+b^2d+bd^2)+abcdr^2s^2+abcds^4$$
$$\geq s^4(a^2c^2+b^2d^2)+s^2abcd(ac+bd)+s^2abcd(as+cs)+sabcd(acs+bds)$$
$$+abcdr^2s^2+abcds^4 \overset{A-G}{\geq} 2s^4abcd+2s^2abcd(ac+bd)+2abcds^4+abcdr^2s^2$$
$$=s^2abcd\left(4s^2+2(ac+bd)+r^2\right) \overset{\text{via }(1)}{=}$$
$$r^2s^4\left(4s^2+2\left(2r^2+2r*\sqrt{4R^2+r^2}\right)+r^2\right) \Rightarrow$$
$$(bs+ca)(as+bd)(ds+ca)(cs+bd) \geq r^2s^4\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right) \to (2)$$

We have : $m^4=$

$$\left(\min\left(\frac{1}{a^{-3}+b^{-3}+c^{-3}},\frac{1}{a^{-3}+b^{-3}+d^{-3}},\frac{1}{a^{-3}+c^{-3}+d^{-3}},\frac{1}{b^{-3}+c^{-3}+d^{-3}}\right)\right)^4$$
$$\leq \frac{(a^3b^3c^3)(a^3b^3d^3)(a^3c^3d^3)(b^3c^3d^3)}{(a^3b^3+b^3c^3+c^3a^3)(a^3b^3+b^3d^3+d^3a^3)(a^3c^3+c^3d^3+d^3a^3)(b^3c^3+c^3d^3+b^3d^3)}$$

$$\le \frac{9^4.F^{18}}{(bs+ca)^3(as+bd)^3(ds+ca)^3(cs+bd)^3}$$

$$\left(\begin{array}{c}\because a^3b^3+b^3c^3+c^3a^3=c^3a^3+b^3(c+a)(c^2-ca+a^2)\\ \ge c^3a^3+\frac{b^3}{4}(c+a)^3=c^3a^3+\frac{b^3s^3}{8}+\frac{b^3s^3}{8}\overset{Holder}{\ge}\frac{\left(\frac{bs}{2}+\frac{bs}{2}+ca\right)^3}{9}\\ \Rightarrow a^3b^3+b^3c^3+c^3a^3\ge\frac{(bs+ca)^3}{9}analogs\end{array}\right)$$

$$\le \frac{9^4.F^{18}}{r^6s^{12}\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right)^3}\overset{?}{\le}\left(\frac{F^2}{3R\sqrt{2}}\right)^4=\frac{F^8}{81R^4.4}$$

$$\Leftrightarrow \frac{9^6.r^{10}s^{10}}{r^6s^{12}\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right)^3}\overset{?}{\le}\frac{1}{4R^4}$$

$$\Leftrightarrow \underbrace{s^2\left(4s^2+4r*\sqrt{4R^2+r^2}+5r^2\right)^3\overset{?}{\ge}9^6.4R^4r^4}_{(*)}$$

Now, via Blundon − Eddy, LHS of $(*)\ge$

$$8r\left(\sqrt{4R^2+r^2}-r\right)\left(4.8r\left(\sqrt{4R^2+r^2}-r\right)+4r*\sqrt{4R^2+r^2}+5r^2\right)^3\overset{?}{\ge}9^6.4R^4r^4$$

$$\Leftrightarrow 2(4R^2+r^2)(256R^2+172r^2)+2r\left(192R^2r+75r^3+96r(4R^2+r^2)\right)-729R^4$$

$$\overset{?}{\ge}2r.\sqrt{4R^2+r^2}.\left(256R^2+172r^2+192R^2+75r^2+96(4R^2+r^2)\right)$$

$$\Leftrightarrow (1319R^4+3040R^2r^2+686r^4)^2\overset{?}{\ge}\left(2r.\sqrt{4R^2+r^2}.(832R^2+343r^2)\right)^2$$

$$\Leftrightarrow 1739761t^3-3056064t^2-849660t+5488\overset{?}{\ge}0\ \left(t=\frac{R^2}{r^2}\right)$$

$$\Leftrightarrow (t-2)\left(1739761t^2+422086t+1372(t-2)\right)\overset{?}{\ge}0\to true\because t\overset{L.Fejes\ Toth}{\ge}2$$

$$\Rightarrow (*)\text{ is true}\Rightarrow m^4\le\left(\frac{F^2}{3R\sqrt{2}}\right)^4$$

$$\therefore \boxed{3R\sqrt{2}.\min\left(\frac{1}{a^{-3}+b^{-3}+c^{-3}},\frac{1}{a^{-3}+b^{-3}+d^{-3}},\frac{1}{a^{-3}+c^{-3}+d^{-3}},\frac{1}{b^{-3}+c^{-3}+d^{-3}}\right)\le F^2}$$

$$\text{Also, }\frac{R\sqrt{2}}{3}.max(a^3+b^3+c^3,a^3+b^3+d^3,a^3+c^3+d^3,b^3+c^3+d^3)\ge$$

$$\frac{R\sqrt{2}}{4.3}(a^3+b^3+c^3+a^3+b^3+d^3+a^3+c^3+d^3+b^3+c^3+d^3)\overset{A-G}{\ge}$$

$$\frac{R\sqrt{2}}{4}(abc+abd+acd+bcd)=\frac{R\sqrt{2}}{4}(acs+bds)\overset{A-G}{\ge}\frac{Rs\sqrt{2}}{4}.2\sqrt{acbd}=$$

$$\frac{Rrs^2}{\sqrt{2}}\overset{?}{\ge}F^2=r^2s^2\Leftrightarrow R\overset{?}{\ge}\sqrt{2}r\to\text{true via L. Fejes Toth}$$

$$\therefore \boxed{F^2\le\frac{R\sqrt{2}}{3}.max(a^3+b^3+c^3,a^3+b^3+d^3,a^3+c^3+d^3,b^3+c^3+d^3)}\ (QED)$$

1472.

In any bicentric quadrilateral ABCD with sides a,b,c,d such that $a\le b\le c\le d$, the following relationship holds :

$$\sqrt[4]{\frac{a}{r}}+\sqrt[4]{\frac{b}{r}}+\sqrt[4]{\frac{c}{r}}\le 3.\sqrt{\frac{R}{r}}$$

Proposed by Emil C. Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$a\le b\le c\le d\Rightarrow b\le d\Rightarrow b\le\frac{b+d}{2}\Rightarrow\sqrt[4]{\frac{a}{r}}+\sqrt[4]{\frac{b}{r}}+\sqrt[4]{\frac{c}{r}}$$

$$\le\frac{1}{\sqrt[4]{r}}\left(\sqrt[4]{a}+\sqrt[4]{c}\right)+\sqrt[4]{\frac{b+d}{2r}}\overset{CBS}{\le}\frac{\sqrt{2}}{\sqrt[4]{r}}.\sqrt{\sqrt{a}+\sqrt{c}}+\sqrt[4]{\frac{b+d}{2r}}\overset{CBS}{\le}$$

$$\frac{\sqrt{2}}{\sqrt[4]{r}}.\sqrt{\sqrt{2}.\sqrt{a+c}}+\sqrt[4]{\frac{b+d}{2r}}=2.\sqrt[4]{\frac{a+c}{2r}}+\sqrt[4]{\frac{b+d}{2r}}=2.\sqrt[4]{\frac{s}{2r}}+\sqrt[4]{\frac{s}{2r}}=3.\sqrt[4]{\frac{s}{2r}}$$

$$\overset{?}{\le}3.\sqrt{\frac{R}{r}}\Leftrightarrow r^2s^2\underset{(*)}{\overset{?}{\le}}4R^4$$

$$\text{Now, via Blundon - Eddy, } s\le\sqrt{4R^2+r^2}+r$$

$$\Rightarrow r^2s^2\le r^2\left(4R^2+2r^2+2r.\sqrt{4R^2+r^2}\right)\overset{?}{\le}4R^4$$

$$\Leftrightarrow 4R^4-4R^2r^2-2r^4\overset{?}{\ge}2r^3.\sqrt{4R^2+r^2}$$

$$\Leftrightarrow\left(4R^4-4R^2r^2-2r^4\right)^2\overset{?}{\ge}4r^6.\left(4R^2+r^2\right)$$

$$\left(\because 4R^4-4R^2r^2-2r^4=\left(4R^2+4r^2\right)\left(R^2-2r^2\right)+6r^4\overset{\text{L.Fejes Toth}}{\ge}6r^4>0\right)$$

$$\Leftrightarrow 16R^6\left(R^2-2r^2\right)\overset{?}{\ge}0\rightarrow\text{true, via L. Fejes Toth}\Rightarrow(*)\text{ is true}$$

$$\therefore\sqrt[4]{\frac{a}{r}}+\sqrt[4]{\frac{b}{r}}+\sqrt[4]{\frac{c}{r}}\le 3.\sqrt{\frac{R}{r}}$$

$\forall$ bicentric quadrilateral ABCD with sides $a,b,c,d \mid a\le b\le c\le d$ (QED)

1473. **In any Δ ABC, the following relationships hold :**

$$2\sum_{cyc}\frac{\sqrt{m_a}}{a}\ge\sum_{cyc}\sqrt{\frac{w_a}{bc-w_a^2}}\ \text{and}\ 4\sum_{cyc}\frac{m_a}{r_a-r}\ge\sum_{cyc}\frac{w_a(r_b+r_c)}{bc-w_a^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$bc-w_a^2=bc-\frac{4bcs(s-a)}{(b+c)^2}=bc-bc.\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$\Rightarrow bc-w_a^2=\frac{a^2bc}{(b+c)^2}\to(1)$$

$$\text{Also, } r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c=4R\cos^2\frac{A}{2}\to(2)$$

$$\text{Now, }\frac{2\sqrt{m_a}}{a}.\sqrt{\frac{bc-w_a^2}{w_a}}\overset{\text{via (1) and Lascu}}{\geq}\frac{2\sqrt{\frac{b+c}{2}\cos\frac{A}{2}}}{a}.\sqrt{\frac{\frac{a^2bc}{(b+c)^2}}{\frac{2bc}{b+c}\cos\frac{A}{2}}}=1$$

$$\Rightarrow\frac{2\sqrt{m_a}}{a}\geq\sqrt{\frac{w_a}{bc-w_a^2}}\text{ and analogs }\therefore 2\sum_{cyc}\frac{\sqrt{m_a}}{a}\geq\sum_{cyc}\sqrt{\frac{w_a}{bc-w_a^2}}$$

$$\text{Again, }\frac{4m_a}{r_a-r}.\frac{bc-w_a^2}{w_a(r_b+r_c)}\overset{\text{via (1),(2) and Lascu}}{\geq}\frac{4.\frac{b+c}{2}\cos\frac{A}{2}}{\frac{rs(s-(s-a))}{s(s-a)}}.\frac{\frac{a^2bc}{(b+c)^2}}{\frac{2bc}{b+c}\cos\frac{A}{2}.4R\frac{s(s-a)}{bc}}=\frac{abc}{4Rrs}$$

$$=1\Rightarrow\frac{4m_a}{r_a-r}\geq\frac{w_a(r_b+r_c)}{bc-w_a^2}\text{ and analogs }\therefore 4\sum_{cyc}\frac{m_a}{r_a-r}\geq\sum_{cyc}\frac{w_a(r_b+r_c)}{bc-w_a^2}\text{ (QED)}$$

1474. In any Δ ABC, the following relationship holds :

$$\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}.\frac{a}{\sqrt{bc-w_a^2}}\leq\sqrt{\frac{2R}{r}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$bc-w_a^2=bc-\frac{4bcs(s-a)}{(b+c)^2}=bc-bc.\frac{(b+c)^2-a^2}{(b+c)^2}=\frac{a^2bc}{(b+c)^2}$$

$$\Rightarrow\frac{a}{\sqrt{bc-w_a^2}}=\frac{b+c}{\sqrt{bc}}\therefore\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}.\frac{a}{\sqrt{bc-w_a^2}}\leq\sqrt{\frac{2R}{r}}$$

$$\Leftrightarrow\frac{2R(4R+r)}{ab+bc+ca}.\frac{(b+c)^2}{bc}\leq\frac{2R}{r}\Leftrightarrow\frac{4R+r}{ab+bc+ca}.\frac{a(b+c)^2}{4Rrs}\leq\frac{1}{r}$$

$$\Leftrightarrow\frac{4R+r}{4R}\leq\frac{s(ab+bc+ca)}{a(b+c)^2}\Leftrightarrow\frac{r}{4R}\leq\frac{s(ab+bc+ca)-a(b+c)^2}{a(b+c)^2}$$

$$\Leftrightarrow \frac{F^2}{sabc} \le \frac{s(ab+bc+ca)-a(b+c)^2}{a(b+c)^2}$$

$$\Leftrightarrow (s-a)(s-b)(s-c)(b+c)^2 \le bc\big(s(ab+bc+ca)-a(b+c)^2\big)$$

$$\Leftrightarrow (z+x)(x+y)\left(\begin{array}{c}\left(\sum_{cyc} x\right)\left(\sum_{cyc}((z+x)(x+y))\right)\\ -(y+z)(2x+y+z)^2\end{array}\right) - xyz(2x+y+z)^2 \ge 0$$

$$(\text{where } x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y)$$

$$\Leftrightarrow x^5 + x^4y + x^4z + xy^2z^2 + y^3z^2 + y^2z^3 \ge 2x^3yz + 2x^2y^2z + 2x^2yz^2 \to \text{true}$$

$$\because x^5 + xy^2z^2 \overset{A-G}{\ge} 2x^3yz, x^4y + y^3z^2 \overset{A-G}{\ge} 2x^2y^2z \text{ and } x^4z + y^2z^3 \overset{A-G}{\ge} 2x^2yz^2$$

$$\therefore \sqrt{\frac{r_a + r_b + r_c}{h_a + h_b + h_c}} \cdot \frac{a}{\sqrt{bc - w_a^2}} \le \sqrt{\frac{2R}{r}} \ \forall \Delta \ ABC \ (QED)$$

1475. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \sqrt[5]{\sin A}\right)\left(\sum_{cyc} \sqrt[5]{\sin \frac{A}{2}}\right) \le \sqrt[10]{\frac{3^{21}}{16}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban

$$f:(0,\pi) \to (0,1], f(x) = \sqrt[5]{\sin x} = (\sin x)^{\frac{1}{5}}$$

$$f'(x) = \frac{1}{5}(\sin x)^{-\frac{4}{5}} \cdot \cos x, f''(x) = -\frac{4}{25}(\sin x)^{-\frac{9}{5}}\cos^2 x - \frac{1}{5}(\sin x)^{-\frac{4}{5}}\sin x < 0$$

$$(\forall) x \in (0,\pi) \Rightarrow f \text{ concave}$$

$$f\left(\frac{A+B+C}{3}\right) \ge \frac{\sum_{cyc} f(A)}{3} \overset{Jensen}{\Rightarrow} \sum_{cyc} f(A) \le 3f(60°) \Rightarrow \sum_{cyc} \sqrt[5]{\sin A} \le 3\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{5}}$$

$$g:(0,\pi) \to (0,1), g(x) = \sqrt[5]{\sin \frac{x}{2}} = \left(\sin \frac{x}{2}\right)^{\frac{1}{5}}$$

$$g'(x) = \frac{1}{5}\left(\sin \frac{x}{2}\right)^{-\frac{4}{5}} \frac{1}{2}\cos \frac{x}{2} = \frac{1}{10}\left(\sin \frac{x}{2}\right)^{-\frac{4}{5}} \cos \frac{x}{2}$$

$$g''(x) = -\frac{4}{50}\left(\sin \frac{x}{2}\right)^{-\frac{9}{5}} \cdot \frac{1}{2} \cdot \cos^2 \frac{x}{2} - \frac{1}{10}\left(\sin \frac{x}{2}\right)^{-\frac{4}{5}} \cdot \frac{1}{2}\sin \frac{x}{2}$$

$$g''(x) < 0, (\forall) x \in (0,\pi) \Rightarrow g \text{ concave} \to \text{Jensen}$$

$$g\left(\frac{A+B+C}{3}\right) \geq \frac{\sum_{cyc} g(A)}{3} \Rightarrow \sum_{cyc} g(A) \leq 3g(60^\circ) \Rightarrow \sum_{cyc} \sqrt[5]{\sin\frac{A}{2}} \leq 3\left(\frac{1}{2}\right)^{\frac{1}{5}}$$

$$\Rightarrow \left(\sum_{cyc} \sqrt[5]{\sin A}\right) \cdot \left(\sum_{cyc} \sqrt[5]{\sin\frac{A}{2}}\right) \leq 9\left(\frac{\sqrt{3}}{4}\right)^{\frac{1}{5}} = \frac{3^{2+\frac{1}{10}}}{4^{\frac{1}{5}}} =$$

$$= \frac{3^{\frac{21}{10}}}{2^{\frac{2}{5}}} = \frac{\sqrt[10]{3^{21}}}{2^{\frac{4}{10}}} = \frac{\sqrt[10]{3^{21}}}{\sqrt[10]{2^4}} = \sqrt[10]{\frac{3^{21}}{16}}$$

$$\Rightarrow \left(\sum_{cyc} \sqrt[5]{\sin A}\right) \cdot \left(\sum_{cyc} \sqrt[5]{\sin\frac{A}{2}}\right) \leq \sqrt[10]{\frac{3^{21}}{16}}$$

Solution 2 by Tapas Das-India

$$\sin^2 A + \sin^2 B + \sin^2 C = \frac{a^2}{4R^2} + \frac{b^2}{4R^2} + \frac{c^2}{4R^2} = \frac{a^2+b^2+c^2}{4R^2}$$

$$\leq \frac{9R^2}{4R^2} = \frac{9}{4}$$

$$\sum \sqrt[5]{\sin A} \overset{Leibniz}{\leq} \sum \sqrt[10]{\sin^2 A} \overset{CBS}{\leq} \frac{3}{3^{10}}\left(\sum \sin^2 A\right)^{\frac{1}{10}} \leq \frac{3}{3^{10}} \cdot \left(\frac{9}{4}\right)^{\frac{1}{10}}$$

$$\text{Let } f(x) = \sin\frac{x}{2}, x \in (0,\pi)$$

$$f'(x) = \frac{1}{2}\cos\frac{x}{2}, \qquad f''(x) = -\frac{1}{4}\sin^2\frac{x}{2} < 0$$

$\therefore f$ **is concave**

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \leq 3f\left(\frac{A+B+C}{6}\right)$$

$$\therefore \sum \sin\frac{A}{2} \qquad = 3 \cdot \sin\frac{\pi}{6} = \frac{3}{2}$$

$$\sum \sqrt[5]{\sin\frac{A}{2}} \overset{CBS}{\leq} \frac{3}{3^{\frac{1}{5}}}\left(\sum\left(\sin\frac{A}{2}\right)\right)^{\frac{1}{5}} = \frac{3}{3^{\frac{1}{5}}}\left(\frac{3}{2}\right)^{\frac{1}{5}} = \frac{3}{2^{\frac{1}{5}}}$$

$$\sum \sqrt[5]{\sin A} \cdot \sum \sqrt[5]{\sin\frac{A}{2}} = \frac{(3^{10})^{\frac{1}{10}}}{(2^2)^{\frac{1}{10}}}$$

$$\le \frac{3}{3^{10}}\left(\frac{9}{4}\right)^{\frac{1}{10}} \cdot \frac{(3^{10})^{\frac{1}{10}}}{(2^2)^{\frac{1}{10}}} = \left(\frac{3^{10}\cdot 3^2\cdot 3^{10}}{3\cdot 2^2\cdot 2^2}\right)^{\frac{1}{10}} = \left(\frac{3^{21}}{16}\right)^{\frac{1}{10}}$$

Solution 3 by Khaled Abd Imouti-Syria

Let be the function:

$$f(x) = \sqrt[5]{\sin x}, x \in]0,\pi[$$

$$f(x) = (\sin x)^{\frac{1}{5}}, 'f(x) = \frac{1}{5}(\sin x)^{-\frac{4}{5}} \cdot \cos x$$

$$''f(x) = \frac{1}{5}\left[\underbrace{-\frac{4}{5}(\sin x)^{-\frac{9}{5}}\cos^2 x}_{<0} - \underbrace{\sin x \cdot (\sin x)^{-\frac{4}{5}}}_{<0}\right] < 0$$

So f is a concave function

$$\sum_{cyc}\sqrt[5]{\sin A} \le \sqrt[5]{\sin\left(\frac{A+B+C}{3}\right)}, \sum_{cyc}\sqrt[5]{\sin A} \le 3\sqrt[5]{\frac{\sqrt{3}}{2}} = 3\,\frac{(3)^{\frac{1}{10}}}{(2)^{\frac{1}{5}}} \quad (*)$$

Let be the function: $g(x) = \sqrt[5]{\sin\frac{x}{2}}, x \in]0,\pi[$

in similar way g is concave function

So: $\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}} \le 3\sqrt[5]{\sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)} \to \sum_{cyc}\sqrt[5]{\sin\frac{A}{2}} \le 3\sqrt[5]{\frac{1}{2}}$ **(**)**

⇒ from (*) and (**)

$$\left(\sum_{cyc}\sqrt[5]{\sin A}\right)\cdot\left(\sum_{cyc}\sqrt[5]{\sin\frac{A}{2}}\right) \le 9\cdot\frac{(3)^{\frac{1}{10}}}{(2)^{\frac{2}{5}}} = \frac{(3)^2\cdot(3)^{\frac{1}{10}}}{(4)^{\frac{1}{5}}} = \frac{(3)^{\frac{21}{10}}}{(4^2)^{\frac{1}{10}}} = \sqrt[10]{\frac{3^{21}}{16}}$$

1476.

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{1+\cot\frac{B}{2}.\cot\frac{C}{2}} \le \frac{3}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{1+\cot\frac{B}{2}.\cot\frac{C}{2}}=\sum_{cyc}\frac{1}{1+\frac{s^2}{r_br_c}}=\sum_{cyc}\frac{s(s-a)}{s(s-a)+s^2}=\sum_{cyc}\frac{2s-a-s}{2s-a}$$

$$=3-\sum_{cyc}\frac{s}{b+c}=3-s.\frac{\sum_{cyc}(\sum_{cyc}ab+a^2)}{2s(s^2+2Rr+r^2)}=3-\frac{(\sum_{cyc}a^2+2\sum_{cyc}ab)+\sum_{cyc}ab}{2(s^2+2Rr+r^2)}$$

$$=3-\frac{4s^2+s^2+4Rr+r^2}{2(s^2+2Rr+r^2)}=\frac{s^2+8Rr+5r^2}{2(s^2+2Rr+r^2)}\overset{?}{\leq}\frac{3}{4}\Leftrightarrow s^2-10Rr-7r^2\overset{?}{\geq}0$$

$$\Leftrightarrow s^2-16Rr+5r^2+6r(R-2r)\overset{?}{\geq}0\to \text{true}\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\geq}0\ and$$

$$6r(R-2r)\overset{\text{Euler}}{\geq}0\therefore\sum_{cyc}\frac{1}{1+\cot\frac{B}{2}.\cot\frac{C}{2}}\leq\frac{3}{4}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1477. In any Δ ABC, the following relationship holds :

$$8\sum_{cyc}\sin^2\frac{A}{2}+32\prod_{cyc}\sin\frac{A}{2}+\frac{9}{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}\geq 22$$

Proposed Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\operatorname{cosec}^2\frac{A}{2}=3+\sum_{cyc}\cot^2\frac{A}{2}=3+\sum_{cyc}\frac{s^2}{r_a^2}$$

$$=3+\frac{s^2}{r^2s^4}\left(\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\left(\sum_{cyc}r_a\right)\right)=3+\frac{s^2\left(s^4-2rs^2(4R+r)\right)}{r^2s^4}$$

$$\Rightarrow\sum_{cyc}\operatorname{cosec}^2\frac{A}{2}=\frac{s^2-8Rr+r^2}{r^2}\Rightarrow\left(\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)^2=\sum_{cyc}\operatorname{cosec}^2\frac{A}{2}+$$

$$2\sum_{cyc}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}=\frac{s^2-8Rr+r^2}{r^2}+\frac{2}{\left(\frac{r}{4R}\right)}.\sum_{cyc}\sin\frac{A}{2}\overset{\text{Jensen}}{\leq}$$

$$\frac{s^2-8Rr+r^2}{r^2}+\frac{12R}{r}\overset{\text{Gerretsen}}{\leq}\frac{4R^2+8Rr+4r^2}{r^2}=\frac{4(R+r)^2}{r^2}$$

$$\Rightarrow\sum_{cyc}\operatorname{cosec}\frac{A}{2}\overset{(1)}{\leq}\frac{2R+2r}{r}$$

$$\text{Now, } 8\sum_{cyc}\sin^2\frac{A}{2}+32\prod_{cyc}\sin\frac{A}{2}+\frac{9}{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}-22$$

$$=\frac{9}{\left(\prod_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)}-22+8\sum_{cyc}\sin^2\frac{A}{2}+32.\frac{r}{4R}\overset{\text{via (1)}}{\geq}$$

$$\frac{9}{\left(\frac{r}{4R}\right)\left(\frac{2R+2r}{r}\right)}-22+\frac{4(2R-r)}{R}+\frac{8r}{R}=\frac{2(2R^2-5Rr+2r^2)}{R}=$$

$$\frac{2(2R-r)(R-2r)}{R}\overset{\text{Euler}}{\geq}0\Rightarrow 8\sum_{cyc}\sin^2\frac{A}{2}+32\prod_{cyc}\sin\frac{A}{2}+\frac{9}{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\geq 22\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1478.

In any Δ ABC, the following relationship holds :

$$4r\left(4+\frac{r}{R}\right)(2R-r)^2\leq\sum_{cyc}r_br_c(r_b+r_c)\leq R(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$4r\left(4+\frac{r}{R}\right)(2R-r)^2\leq\sum_{cyc}r_br_c(r_b+r_c)$$

$$\Leftrightarrow rs^2\sum_{cyc}\frac{r_b+r_c}{r_a}\geq r.\frac{4\sum_{cyc}r_a}{4R+r-\frac{rs^2}{s^2}}.\left(4R+r-\frac{rs^2}{s^2}-2.\frac{rs^2}{s^2}\right)^2$$

$$\Leftrightarrow\left(\sum_{cyc}r_ar_b\right)\left(\sum_{cyc}\frac{r_a(r_b^2+r_c^2)}{r_ar_br_c}\right)\geq\frac{4\sum_{cyc}r_a}{\sum_{cyc}r_a-\frac{r_ar_br_c}{\sum_{cyc}r_ar_b}}.\left(\sum_{cyc}r_a-\frac{3r_ar_br_c}{\sum_{cyc}r_ar_b}\right)^2$$

$$\Leftrightarrow\left(\sum_{cyc}r_ar_b\right)\left(\sum_{cyc}\frac{r_ar_b\left(\sum_{cyc}r_a-r_c\right)}{r_ar_br_c}\right)$$

$$\geq\frac{4\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-3r_ar_br_c\right)^2}{\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-r_ar_br_c\right)\left(\sum_{cyc}r_ar_b\right)^2}$$

$$\Leftrightarrow\left(\sum_{cyc}r_ar_b\right)\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-3r_ar_br_c\right)\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-r_ar_br_c\right)\left(\sum_{cyc}r_ar_b\right)$$

$$\geq 4r_a r_b r_c\left(\sum_{cyc} r_a\right)\left(\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_a r_b\right)-3r_a r_b r_c\right)^2$$

$$\Leftrightarrow\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)-xyz\right)\left(\sum_{cyc} xy\right)^2 \overset{(*)}{\geq} 4xyz\left(\sum_{cyc} x\right)\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)-3xyz\right)$$

$(x = r_a, y = r_b, z = r_c)$

Assigning $y+z = X, z+x = Y, x+y = Z \Rightarrow X+Y-Z = 2z > 0, Y+Z-X = 2x > 0$ and $Z+X-Y = 2y > 0 \Rightarrow X+Y > Z, Y+Z > X, Z+X > Y$
$\Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s', R', r'$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s' \Rightarrow \sum_{cyc} x = s' \to (1) \Rightarrow x = s' - X$,
$y = s' - Y, z = s' - Z$ and such substitutions $\Rightarrow xyz = r'^2 s' \to (2)$ and
$$\sum_{cyc} xy = \sum_{cyc}(s'-X)(s'-Y) \Rightarrow \sum_{cyc} xy = 4R'r' + r'^2 \to (3)$$
$\therefore$ via $(1), (2)$ and $(3), (*) \Leftrightarrow$
$$\left(s'\left(4R'r'+r'^2\right)-r'^2 s'\right)\left(4R'r'+r'^2\right)^2 \geq 4r'^2 s'(s')\left(s'\left(4R'r'+r'^2\right)-r'^2 s'\right)$$
$\Leftrightarrow R'(4R'+r')^2 \geq (4R'-2r')s'^2 \to$ true via Blundon – Gerretsen $\Rightarrow (*)$ is true
$$\therefore \boxed{4r\left(4+\frac{r}{R}\right)(2R-r)^2 \leq \sum_{cyc} r_b r_c (r_b + r_c)}$$
$$\Rightarrow \sum_{cyc} r_b r_c\left(\sum_{cyc} r_a - r_a\right) \geq \frac{4r}{R}(4R+r)(2R-r)^2 \Rightarrow (4R+r)s^2 - 3rs^2$$
$$\geq \frac{4r}{R}(4R+r)(2R-r)^2 \Rightarrow Rs^2 \geq r(4R+r)(4R-2r)$$
$$\Rightarrow \boxed{R's'^2 \overset{(\bullet)}{\geq} r'(4R'+r')(4R'-2r')}$$
Again, $\sum_{cyc} r_b r_c (r_b + r_c) \leq R(4R+r)^2$
$$\Leftrightarrow 4xyz\sum_{cyc}\frac{y+z}{x} \leq \left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x - \frac{xyz}{\sum_{cyc} xy}\right)$$
$$\Leftrightarrow s'^2\left(s'\left(4R'r'+r'^2\right)-r'^2 s'\right) \geq 4\left(4R'r'+r'^2\right)\left(s'\left(4R'r'+r'^2\right)-3r'^2 s'\right) \Leftrightarrow$$
$R's'^2 \geq r'(4R'+r')(4R'-2r') \to$ true via $(\bullet) \therefore \boxed{\sum_{cyc} r_b r_c (r_b + r_c) \leq R(4R+r)^2}$
$$\Rightarrow 4r\left(4+\frac{r}{R}\right)(2R-r)^2 \leq \sum_{cyc} r_b r_c (r_b + r_c) \leq R(4R+r)^2$$
$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1479. In any Δ ABC, the following relationship holds :

$$\sqrt[3]{m_a m_b m_c}+\frac{|m_a-m_b|+|m_b-m_c|+|m_c-m_a|}{2}\ge\frac{m_a+m_b+m_c}{3}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, we shall prove : }\frac{|a-b|+|b-c|+|c-a|}{2}\overset{(1)}{\ge}\frac{a+b+c}{3}-\sqrt[3]{abc}$$

$$\text{Now, }\frac{a+b+c}{3}-\sqrt[3]{abc}\overset{G-H}{\le}\frac{\sum_{cyc}a}{3}-\frac{3abc}{\sum_{cyc}ab}$$

$$=\frac{(\sum_{cyc}a)(\sum_{cyc}ab)-9abc}{3\sum_{cyc}ab}\overset{?}{\le}\frac{|a-b|+|b-c|+|c-a|}{2}$$

$$\Leftrightarrow\sum_{cyc}(b-c)^2+2\sum_{cyc}(|a-b||b-c|)\underset{(*)}{\overset{?}{\ge}}\frac{4}{9}\left(\frac{(\sum_{cyc}a)(\sum_{cyc}ab)-9abc}{\sum_{cyc}ab}\right)^2$$

$$\text{Now, LHS of }(*)\ge 2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\overset{?}{\ge}\frac{4}{9}\left(\frac{(\sum_{cyc}a)(\sum_{cyc}ab)-9abc}{\sum_{cyc}ab}\right)^2$$

$$\Leftrightarrow 9(s^2-12Rr-3r^2)(s^2+4Rr+r^2)^2\overset{?}{\ge}2\left(2s(s^2+4Rr+r^2)-36Rrs\right)^2$$

$$\Leftrightarrow s^6+(188Rr-25r^2)s^4-r^2s^2(2288R^2+136Rr+53r^2)-27r^3(4R+r)^3\underset{(**)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)\overset{Gerretsen}{\ge}(204Rr-30r^2)s^4-r^2s^2(2288R^2+136Rr+53r^2)$$

$$-27r^3(4R+r)^3\overset{Gerretsen}{\ge}\begin{pmatrix}(204Rr-30r^2)(16Rr-5r^2)\\-r^2(2288R^2+136Rr+53r^2)\end{pmatrix}s^2-27r^3(4R+r)^3$$

$$=r^2(976R^2-1636Rr+97r^2)s^2-27r^3(4R+r)^3\overset{Gerretsen}{\ge}$$

$$r^3(976R^2-1636Rr+97r^2)(16R-5r)-27r^3(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow 434t^3-1011t^2+294t-16\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(362t^2+72t(t-2)+t+8\right)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a+b+c}{3}-\sqrt[3]{abc}\le\frac{|a-b|+|b-c|+|c-a|}{2}\Rightarrow(1)\text{ is true and invoking }(1)$$

on a triangle with sides m_a, m_b, m_c, we arrive at :

$$\frac{|m_a-m_b|+|m_b-m_c|+|m_c-m_a|}{2}\ge\frac{m_a+m_b+m_c}{3}-\sqrt[3]{m_a m_b m_c}$$

$$\Rightarrow\sqrt[3]{m_a m_b m_c}+\frac{|m_a-m_b|+|m_b-m_c|+|m_c-m_a|}{2}\ge\frac{m_a+m_b+m_c}{3}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

1480. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\ge\sum_{cyc}\frac{a}{b}+9g_ag_bg_c$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality}\Rightarrow g_a\le AI+r\overset{?}{\le}w_a\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}}\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}\frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}+\frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}}\Leftrightarrow(b+c)\sin\frac{A}{2}\overset{?}{\le}a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}\overset{?}{\le}4R\sin\frac{A}{2}\cos\frac{A}{2}\Leftrightarrow\cos\frac{B-C}{2}\overset{?}{\le}1\to\text{true}$$

$$\therefore g_a\le w_a\overset{A-G}{\le}\sqrt{s(s-a)}\text{ and analogs}$$

$$\Rightarrow 9g_ag_bg_c\le\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}\therefore 9g_ag_bg_c\overset{(\bullet)}{\le}9rs^2$$

$$\text{Now, }\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\ge\sum_{cyc}\frac{a}{b}+9g_ag_bg_c$$

$$\Leftrightarrow\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\ge\sum_{cyc}\frac{a+b+c-(b+c)}{b}+9g_ag_bg_c$$

$$\Leftrightarrow\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}\ge\frac{2s}{4Rrs}\sum_{cyc}ab+9g_ag_bg_c$$

$$\Leftrightarrow\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}\overset{(*)}{\ge}\frac{1}{2Rr}\sum_{cyc}ab+9g_ag_bg_c$$

$$\text{We have : }\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}\ge\sum_{cyc}\frac{s(s-a)\sqrt{bc}\cos\frac{A}{2}\tan\frac{A}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}\overset{G-H}{\ge}$$

$$\sum_{cyc}\frac{s(s-a).\frac{2bc}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}.\sin\frac{A}{2}}{\frac{r}{4R}.\frac{4R}{s}}\overset{\substack{0<\cos\frac{B-C}{2}\le1\\ \text{and}\\ \because s\tan\frac{A}{2}=r_a}}{\ge}\sum_{cyc}\frac{s(s-a)h_ar_a}{r}\overset{A-G}{\ge}\frac{3s}{r}.\sqrt[3]{r^2s.\frac{2r^2s^2}{R}.rs^2}\overset{?}{\ge}9rs^2$$

$$\Leftrightarrow\frac{r^2s.\frac{2r^2s^2}{R}.rs^2}{r^3}\overset{?}{\ge}27r^3s^3\Leftrightarrow 2s^2\overset{?}{\ge}27Rr\to\text{true}\because 2s^2\overset{\text{Gerretsen}}{\ge}$$

$$27Rr+5r(R-2r)\overset{\text{Euler}}{\geq}27Rr\therefore\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}\geq 9rs^2\overset{\text{via }(\bullet)}{\geq}9g_ag_bg_c\to(1)$$

$$\text{Again},\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}-\frac{1}{2Rr}\sum_{cyc}ab\overset{\substack{A-G\\ and\\ Gerretsen}}{\geq}\frac{3R^2}{4r^2}+6-\frac{4R^2+8Rr+4r^2}{2Rr}$$

$$=\frac{3R^2+24r^2}{4r^2}-\frac{4R^2+8Rr+4r^2}{2Rr}\overset{?}{\geq}0\Leftrightarrow 3t^3-8t^2+8t-8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(2t^2+t(t-2)+4\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\therefore\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}\geq\frac{1}{2Rr}\sum_{cyc}ab\to(2)$$

$$\therefore(1)+(2)\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\geq\sum_{cyc}\frac{a}{b}+9g_ag_bg_c\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1481. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}+\frac{R^{n+1}}{r^{n+1}}\geq 2^{n+1}+\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}-\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}=\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}.1.1.1}-\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}.1.1}$$

$$\overset{\substack{G-H\\ and\\ A-G}}{\geq}\sum_{cyc}\frac{4}{\frac{b^3+c^3}{2a^3}+1+1+1}-\sum_{cyc}\frac{\frac{2a^2}{b^2+c^2}+1+1}{3}$$

$$=\sum_{cyc}\frac{8a^3}{b^3+c^3+6a^3}-\frac{2}{3}\left(\sum_{cyc}a^2\right)\sum_{cyc}\frac{1}{b^2+c^2}\overset{\substack{Holder\\ and\\ A-G\text{ and Leibnitz}}}{\geq}$$

$$\frac{8(2s)^3}{3\left(2\sum_{cyc}a^3+6\sum_{cyc}a^3\right)}-\frac{2}{3}.(9R^2).\sum_{cyc}\frac{a}{2abc}$$

$$=\frac{4s^2}{3(s^2-6Rr-3r^2)}-\frac{2}{3}.(9R^2).\frac{2s}{8Rrs}=\frac{4s^2}{3(s^2-6Rr-3r^2)}-\frac{3R}{2r}\overset{?}{\geq}-\frac{R^3-8r^3}{r^3}$$

$$\Leftrightarrow(6R^3-9Rr^2-40r^3)s^2\underset{(*)}{\overset{?}{\geq}}r(36R^4+18R^3r-54R^2r^2-315Rr^3-144r^4)$$

$\boxed{Case\ 1}\ 6R^3 - 9Rr^2 - 40r^3 \geq 0$ *and then* : LHS of $(*) \overset{\text{Gerretsen}}{\geq}$

$$(6R^3 - 9Rr^2 - 40r^3)(16Rr - 5r^2) \overset{?}{\geq} r\begin{pmatrix}36R^4 + 18R^3r - 54R^2r^2 \\ -315Rr^3 - 144r^4\end{pmatrix}$$

$$\Leftrightarrow 30t^4 - 24t^3 - 45t^2 - 140t + 172 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(30t^2 + 96t + 219) + 352\right) \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$\boxed{Case\ 2}\ 6R^3 - 9Rr^2 - 40r^3 < 0$ *and then* : LHS of $(*) = -(40r^3 + 9Rr^2 - 6R^3)s^2$

$$\overset{\text{Gerretsen}}{\geq} -(40r^3 + 9Rr^2 - 6R^3)(4R^2 + 4Rr + 3r^2)$$

$$\overset{?}{\geq} r(36R^4 + 18R^3r - 54R^2r^2 - 315Rr^3 - 144r^4)$$

$$\Leftrightarrow 12t^5 - 6t^4 - 18t^3 - 71t^2 + 64t + 12 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(12t^3 + 42t^2 + 102t + 169) + 332\right) \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (*)$ is true *and* combining *cases* 1 *and* 2, $(*)$ is true $\forall \Delta ABC$

$$\therefore \sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3 + c^3}} - \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2 + c^2}} \geq -\frac{R^3 - 8r^3}{r^3} \to (1)$$

Let $f(n) = t^{n+1} - 2^{n+1}\ \forall t = \frac{R}{r} \geq 2$ *and* $\forall n \geq 2$ *and* then :

$$f'(n) = t^{n+1}.(\ln t) - 2^{n+1}.(\ln 2) \geq 0 \because t^{n+1} \geq 2^{n+1} \text{ and } \ln t \geq \ln 2$$

$$\Rightarrow t^{n+1}.(\ln t) - 2^{n+1}.(\ln 2) \geq 0 \therefore f(n) \text{ is} \uparrow \forall n \geq 2 \Rightarrow f(n) \geq f(2) = \left(\frac{R}{r}\right)^3 - 8$$

$$\Rightarrow \frac{R^3 - 8r^3}{r^3} \leq \left(\frac{R}{r}\right)^{n+1} - 2^{n+1} \therefore -\frac{R^3 - 8r^3}{r^3} \geq -\left(\frac{R^{n+1}}{r^{n+1}} - 2^{n+1}\right) \to (2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3 + c^3}} - \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2 + c^2}} \geq -\left(\frac{R^{n+1}}{r^{n+1}} - 2^{n+1}\right)$$

$$\therefore \sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3 + c^3}} + \frac{R^{n+1}}{r^{n+1}} \geq 2^{n+1} + \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2 + c^2}}$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1482.

In *any* ΔABC *and* $\forall n \geq 2$, the following relationship holds :

$$\min\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}},\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}+\left(\frac{R}{2r}\right)^{n+1}$$

$$\geq 1+max\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}},\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Firstly, we shall prove : $\boxed{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}+\frac{R^3-8r^3}{8r^3}\overset{(i)}{\geq}\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}}$

$$\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}=\sum_{cyc}\sqrt[4]{\frac{2a}{b+c}\cdot\frac{a^2}{b^2-bc+c^2}\cdot 1.1}\overset{G-H}{\geq}\sum_{cyc}\frac{4}{\frac{b+c}{2a}+\frac{b^2-bc+c^2}{a^2}+2}$$

$$=\sum_{cyc}\frac{8a^2}{a(b+c)+2b^2-2bc+2c^2+4a^2}\overset{Bergstrom}{\geq}$$

$$\frac{8.4s^2}{2\sum_{cyc}ab+2\sum_{cyc}a^2-2\sum_{cyc}ab+2\sum_{cyc}a^2+4\sum_{cyc}a^2}=\frac{4s^2}{\sum_{cyc}a^2}$$

$$\therefore\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}\overset{(1)}{\geq}\frac{2s^2}{s^2-4Rr-r^2}$$

$$\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}=\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}.1.1}\overset{A-G}{\leq}\sum_{cyc}\frac{\frac{2a^2}{b^2+c^2}+2}{3}=\frac{2}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{b^2+c^2}\right)$$

$$\overset{A-G}{\leq}\frac{2}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{a}{2abc}\right)=\frac{2.2s}{3.8Rrs}\left(\sum_{cyc}a^2\right)\therefore\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\overset{(2)}{\leq}\frac{s^2-4Rr-r^2}{3Rr}$$

$\therefore$ (1), (2) $\Rightarrow$ in order to prove (i), it suffices to prove :

$$\frac{2s^2}{s^2-4Rr-r^2}+\frac{R^3-8r^3}{8r^3}\overset{(\blacksquare)}{\geq}\frac{s^2-4Rr-r^2}{3Rr}$$

$$\Leftrightarrow\frac{(R^3-8r^3)(s^2-4Rr-r^2)+16r^3s^2}{8r^3(s^2-4Rr-r^2)}\geq\frac{s^2-4Rr-r^2}{3Rr}$$

$$\Leftrightarrow 8r^2s^4-(3R^4+88Rr^3+16r^4)s^2$$

$$+r(12R^5+3R^4r+32R^2r^3+40Rr^4+8r^5)\overset{(*)}{\leq}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where

$m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$

$$\Rightarrow 8r^2s^4-8r^2s^2(4R^2+20Rr-2r^2)+8r^3(4R+r)^3\le 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove}: 8r^2s^4-(3R^4+88Rr^3+16r^4)s^2$$
$$+r(12R^5+3R^4r+32R^2r^3+40Rr^4+8r^5)$$
$$\le 8r^2s^4-8r^2s^2(4R^2+20Rr-2r^2)+8r^3(4R+r)^3$$

$$\Leftrightarrow (3R^4-32R^2r^2-72Rr^3+32r^4)s^2\overset{(**)}{\ge} Rr\begin{pmatrix}12R^4+3R^3r-512R^2r^2\\-352Rr^3-56r^4\end{pmatrix}$$

$\boxed{Case\ 1}$ $3R^4-32R^2r^2-72Rr^3+32r^4\ge 0$ *and* then, LHS of $(**)$

$$\overset{\text{Gerretsen}}{\ge}(3R^4-32R^2r^2-72Rr^3+32r^4)(16Rr-5r^2)$$
$$\overset{?}{\ge} Rr(12R^4+3R^3r-512R^2r^2-352Rr^3-56r^4)$$
$$\Leftrightarrow 18t^5-9t^4-320t^2+464t-80\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big((t-2)(18t^3+63t^2+180t+148)+336\Big)\to \text{true} \because t\overset{\text{Euler}}{\ge}2$$
$$\Rightarrow (**)\text{ is true}$$

$\boxed{Case\ 2}$ $3R^4-32R^2r^2-72Rr^3+32r^4<0$ *and* then, LHS of $(**)\overset{\text{Rouche}}{\ge}$

$$-\Big(-(3R^4-32R^2r^2-72Rr^3+32r^4)\Big)\Big(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\Big)$$
$$\overset{?}{\ge} Rr(12R^4+3R^3r-512R^2r^2-352Rr^3-56r^4)$$
$$\Leftrightarrow 2(R-2r)(3R^5+15R^4r-5R^3r^2+14R^2r^3-108Rr^4+8r^5)\overset{(***)}{\ge}$$
$$2\Big(-(3R^4-32R^2r^2-72Rr^3+32r^4)\Big)(R-2r).\sqrt{R^2-2Rr}$$

$$\text{Now, } 3R^5+15R^4r-5R^3r^2+14R^2r^3-108Rr^4+8r^5$$
$$=(R-2r)(3R^4+21R^3r+37R^2r^2+88Rr^3+68r^4)+144r^5\overset{\text{Euler}}{\ge}144r^5>0$$
$$\therefore \text{in order to prove } (***), \text{it suffices to prove}:$$
$$(3R^5+15R^4r-5R^3r^2+14R^2r^3-108Rr^4+8r^5)^2$$
$$>(R^2-2Rr)(3R^4-32R^2r^2-72Rr^3+32r^4)^2\ \left(\because R-2r\overset{\text{Euler}}{\ge}0\right)$$
$$\Leftrightarrow 108t^9+387t^8-18t^7-2283t^6-5508t^5+7596t^4$$
$$+7776t^3+1648t^2+320t+64>0\Leftrightarrow$$
$$(t-2)\left((t-2)\begin{pmatrix}108t^7+819t^6+2826t^5+5745t^4+\\6168t^3+9288t^2+20256t+45520\end{pmatrix}+101376\right)+20736>0$$
$$\to \text{true} \because t\overset{\text{Euler}}{\ge}2\Rightarrow(***)\Rightarrow(**)\text{ is true}$$
$$\therefore \text{combining } cases\ 1\ and\ 2, (**)\Rightarrow(*)\Rightarrow(\text{i})\text{ is true }\forall\,\Delta\,ABC$$

We sh*all* now prove : $\boxed{\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}+\frac{R^3-8r^3}{8r^3}\overset{(\text{ii})}{\ge}\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}}$

We h*a*ve : $\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}\overset{A-G}{\le}\sum_{cyc}\sqrt[4]{\frac{2a^3.2}{(b^2+c^2)(b+c)}}=\sum_{cyc}\sqrt[4]{\frac{2a^2}{b^2+c^2}.\frac{2a}{b+c}.1.1}$

$$\overset{A-G}{\le}\frac{1}{4}\left(\sum_{cyc}\frac{2a^2}{b^2+c^2}+2+2\sum_{cyc}\frac{a}{b+c}\right)\overset{A-G}{\le}\frac{1}{2}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{a}{2abc}\right)+\frac{1}{2}\sum_{cyc}\frac{a}{b+c}$$

$$\therefore \sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}} \overset{(3)}{\leq} \frac{s^2-4Rr-r^2}{4Rr} + \frac{1}{2}\sum_{cyc}\frac{a}{b+c} \text{ and } \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}$$

$$= \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}.1.1} \overset{G-H}{\geq} \sum_{cyc}\frac{3}{\frac{b^2+c^2}{2a^2}+2} = \sum_{cyc}\frac{6a^2}{b^2+c^2+4a^2} \overset{Bergstrom}{\geq}$$

$$\frac{6.4s^2}{2\sum_{cyc}a^2+4\sum_{cyc}a^2} \therefore \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}} \overset{(4)}{\geq} \frac{2s^2}{s^2-4Rr-r^2} \therefore (3),(4) \Rightarrow$$

in order to prove (ii), it suffices to prove :

$$\frac{2s^2}{s^2-4Rr-r^2} + \frac{R^3-8r^3}{8r^3} \geq \frac{s^2-4Rr-r^2}{4Rr} + \frac{1}{2}\sum_{cyc}\frac{a}{b+c} \text{ and } \because \text{ via } (\blacksquare),$$

$$\frac{2s^2}{s^2-4Rr-r^2} + \frac{R^3-8r^3}{8r^3} \geq \frac{s^2-4Rr-r^2}{3Rr} \therefore \text{it suffices to prove :}$$

$$\frac{s^2-4Rr-r^2}{3Rr} \geq \frac{s^2-4Rr-r^2}{4Rr} + \frac{1}{2}\sum_{cyc}\frac{a}{b+c} \Leftrightarrow \frac{s^2-4Rr-r^2}{12Rr} \geq \frac{1}{2}\sum_{cyc}\frac{2s-(b+c)}{b+c}$$

$$\Leftrightarrow \frac{s^2-4Rr-r^2}{6Rr} \geq \frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} - 3$$

$$\Leftrightarrow s^4 - 14Rrs^2 + r^2(4R^2+6Rr-r^2) \overset{(\bullet)}{\geq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{Gerretsen}{\geq} (2Rr-5r^2)s^2 + r^2(4R^2+6Rr-r^2)$$

$$= (2Rr-4r^2)s^2 - r^2s^2 + r^2(4R^2+6Rr-r^2) \overset{Gerretsen}{\geq}$$

$$(2Rr-4r^2)(16Rr-5r^2) - r^2(4R^2+4Rr+3r^2) + r^2(4R^2+6Rr-r^2)$$

$$= 8r^2(R-2r)(4R-r) \overset{Euler}{\geq} 0 \Rightarrow (\bullet) \Rightarrow \text{(ii) is true} \therefore \text{(i) and (ii)} \Rightarrow$$

$$\min\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\} + \frac{R^3-8r^3}{8r^3}$$

$$\geq \max\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

$$\Rightarrow \boxed{\frac{R^3-8r^3}{8r^3} \overset{(iii)}{\geq} \max\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\} - \min\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}}$$

$$\text{Now, } n \geq 2 \text{ and Euler} \Rightarrow (n+1-3).\ln\frac{R}{2r} \geq 0 \Rightarrow \left(\frac{R}{2r}\right)^{n+1} \geq \left(\frac{R}{2r}\right)^3$$

$$\Rightarrow \left(\frac{R}{2r}\right)^{n+1} - 1 \geq \frac{R^3-8r^3}{8r^3} \overset{\text{via (iii)}}{\geq}$$

$$\max\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\} - \min\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

$$\therefore \min\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}},\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}+\left(\frac{R}{2r}\right)^{n+1}$$

$$\geq 1+max\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}},\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

$\forall\,\Delta$ ABC and $\forall$ n ≥ 2," = " iff Δ ABC is equilateral (QED)

1483.

Let $n \geq 2$. Then, in Δ ABC, the following relationship holds :

$$\sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\left(\frac{R}{r}\right)^n \geq 2^n+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{b+c-a}=\frac{1}{2}\sum_{cyc}\frac{a-s+s}{s-a}=\frac{1}{2}\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)=\frac{2R-r}{r}$$

$$\Rightarrow \sum_{cyc}\frac{a}{b+c-a}=\frac{2R}{r}-1 \rightarrow (1)$$

Invoking (1) on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$, whose area via elementary calculations $=\frac{F}{3}$, we arrive at : $\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}$

$$=2.\frac{\frac{2m_a}{3}.\frac{2m_b}{3}.\frac{2m_c}{3}}{\frac{4F}{3}}.\frac{\frac{\frac{2m_a}{3}+\frac{2m_b}{3}+\frac{2m_c}{3}}{2}}{\frac{F}{3}}-1=\frac{4m_am_bm_c(m_a+m_b+m_c)}{9r^2s^2}-1$$

$$\overset{m_am_bm_c\leq\frac{Rs^2}{2}\text{ and } m_a+m_b+m_c\leq 4R+r}{\leq}\frac{2Rs^2(4R+r)}{9r^2s^2}-1 \Rightarrow \sum_{cyc}\frac{m_a}{m_b+m_c-m_a}\leq\frac{8R^2+2Rr-9r^2}{9r^2}\rightarrow(2)$$

(1) and (2) $\Rightarrow$ in order to prove : $\sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\frac{R^2-4r^2}{4r^2}\geq$

$\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$, it suffices to prove :

$$\sqrt{\frac{2R-r}{r}}+\frac{R^2-4r^2}{4r^2}\geq\sqrt{\frac{8R^2+2Rr-9r^2}{9r^2}}\Leftrightarrow\frac{R^2-4r^2}{4r^2}\geq\frac{\frac{8R^2+2Rr-9r^2}{9r^2}-\frac{2R-r}{r}}{\sqrt{\frac{8R^2+2Rr-9r^2}{9r^2}}+\sqrt{\frac{2R-r}{r}}}$$

$$\Leftrightarrow\frac{(R-2r)(R+2r)}{4r^2}\overset{(*)}{\geq}\frac{3r.8R(R-2r)}{9r^2\left(\sqrt{8R^2+2Rr-9r^2}+\sqrt{9r(2R-r)}\right)}$$

$$8R^2+2Rr-9r^2-27r^2=2(R-2r)(4R+9r)\overset{\text{Euler}}{\geq}0$$

$$\Rightarrow\sqrt{8R^2+2Rr-9r^2}\overset{(i)}{\geq}\sqrt{27r^2}$$

$$\text{and } 9r(2R-r)\overset{\text{Euler}}{\geq}9r(4r-r)\Rightarrow\sqrt{9r(2R-r)}\overset{(ii)}{\geq}\sqrt{27r^2}$$

$$\text{Now, }\because R-2r\overset{\text{Euler}}{\geq}0\therefore\text{ in order to prove }(*)\text{, it suffices to prove :}$$

$$\frac{R+2r}{4}>\frac{8Rr}{3\left(\sqrt{8R^2+2Rr-9r^2}+\sqrt{9r(2R-r)}\right)}$$

$$\Leftrightarrow 9(R+2r)^2\left(\sqrt{8R^2+2Rr-9r^2}+\sqrt{9r(2R-r)}\right)^2>1024R^2r^2$$

$$\Leftrightarrow 9(R+2r)^2\begin{pmatrix}8R^2+2Rr-9r^2+9r(2R-r)\\+2.\sqrt{8R^2+2Rr-9r^2}.\sqrt{9r(2R-r)}\end{pmatrix}\overset{(**)}{>}1024R^2r^2$$

$$\text{Again, via (i) and (ii), LHS of }(**)\geq$$

$$9(R+2r)^2\left(8R^2+2Rr-9r^2+9r(2R-r)+54r^2\right)\overset{?}{>}1024R^2r^2$$

$$\Leftrightarrow 72R^4+468R^3r+308R^2r^2+2016Rr^3+1296r^4\overset{?}{>}0\to\text{true}$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\Rightarrow\sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\frac{R^2-4r^2}{4r^2}\geq\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$$

$$\Rightarrow\frac{R^2-4r^2}{4r^2}\geq\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}-\sqrt{\sum_{cyc}\frac{a}{b+c-a}}\to(3)$$

$$\text{Let } f(n)=t^n-2^n\ \forall\ t=\frac{R}{r}\geq 2\ (t\to\text{fixed})\text{ and }\forall\ n\geq 2\text{ and then :}$$

$$f'(n)=t^n.(\ln t)-2^n.(\ln 2)\geq 0\because t^n\geq 2^n\text{ and }\ln t\geq\ln 2\Rightarrow t^n.(\ln t)-2^n.(\ln 2)$$

$$\geq 0\therefore f(n)\text{ is}\uparrow\forall\ n\geq 2\Rightarrow f(n)\geq f(2)=\left(\frac{R}{r}\right)^2-4\Rightarrow\left(\frac{R}{r}\right)^n-2^n\geq\frac{R^2-4r^2}{4r^2}$$

$$\overset{\text{via }(3)}{\geq}\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}-\sqrt{\sum_{cyc}\frac{a}{b+c-a}}$$

$$\therefore\sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\left(\frac{R}{r}\right)^n\geq 2^n+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$$

$$\forall\ \Delta ABC\text{ and }\forall\ n\geq 2,\text{"="}\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Proof of $m_a m_b m_c\leq\frac{Rs^2}{2}$

$$m_a^2m_b^2m_c^2=\frac{1}{64}\left(2b^2+2c^2-a^2\right)\left(2c^2+2a^2-b^2\right)\left(2a^2+2b^2-c^2\right)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now,}\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\overset{(3)}{=}$$

$$\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\ \therefore (1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\left(\begin{matrix}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{matrix}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\le\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \overset{(\bullet)}{\leq} 0$$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$
$-r^3(4R + r)^3 \overset{?}{\leq} 0$
$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \underbrace{\overset{?}{\geq}}_{(\bullet\bullet)} 20rs^4$

Now, LHS of $(\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)} s^2(16Rr - 5r^2)(8R - 16r)$
$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$ *and*
RHS of $(\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2(4R^2 + 4Rr + 3r^2)$
$(a), (b) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :
$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$
$\geq 20rs^2(4R^2 + 4Rr + 3r^2)$
$\Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0$
$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$
Now, LHS of $(\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3$
and RHS of $(\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2(4R^2 + 4Rr + 3r^2)$
$(c), (d) \Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :
$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$
$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \left(\text{where } t = \frac{R}{r}\right)$
$\Leftrightarrow (t - 2)((t - 2)(224t + 309) + 648) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$
$\Rightarrow (\bullet)$ is true $\Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2}$ (QED)

1484. Let $n \geq 2$. In ΔABC the following relationship holds

$$\sum_{cyc} \sqrt[3]{\frac{a+b}{a+b-c}} + \left(\frac{R}{r}\right)^n \geq 2^n + \sum_{cyc} \sqrt[3]{\frac{m_a + m_b}{m_a + m_b - m_c}}.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sum_{cyc} \sqrt[3]{\frac{a+b}{a+b-c}} \geq 3\sqrt[9]{\frac{(a+b)(b+c)(c+a)}{(a+b-c)(b+c-a)(c+a-b)}} \overset{\substack{Cesaro\\Padoa}}{\geq} 3\sqrt[9]{\frac{8abc}{abc}} = 3\sqrt[3]{2}.$$

Since $(m_a+m_b+m_c)(m_a+m_b-m_c)(m_a-m_b+m_c)(-m_a+m_b+m_c)$
$=9F^2$, then

$$\sum_{cyc}\sqrt[3]{\frac{m_a+m_b}{m_a+m_b-m_c}}$$

$$=\sum_{cyc}\sqrt[3]{\frac{(m_a+m_b+m_c)(m_a+m_b)(m_b+m_c-m_a)(m_c+m_a-m_b)}{9F^2}}$$

$$\overset{AM-GM}{\leq}\sqrt[3]{\frac{m_a+m_b+m_c}{9F^2.4}}.\sum_{cyc}\frac{(m_a+m_b)+2(m_b+m_c-m_a)+2(m_c+m_a-m_b)}{3}$$

$$=\sqrt[3]{\frac{2(m_a+m_b+m_c)^4}{9s^2r^2}}\overset{\substack{Gotman\\Mitrinovic}}{\leq}\sqrt[3]{\frac{2\left(\frac{9R}{2}\right)^4}{9.27r^2.r^2}}=\frac{3}{2}\sqrt[3]{\left(\frac{R}{r}\right)^4}.$$

So it suffices to prove that

$$3\sqrt[3]{2}+\left(\frac{R}{r}\right)^n\geq 2^n+\frac{3}{2}\sqrt[3]{\left(\frac{R}{r}\right)^4}\quad\text{or}\quad\sqrt[3]{\left(\frac{R}{r}\right)^4}.\left(\left(\frac{R}{r}\right)^{n-\frac{4}{3}}-\frac{3}{2}\right)\geq 2^n-3\sqrt[3]{2},$$

which is true because, $\sqrt[3]{\left(\frac{R}{r}\right)^4}\geq\sqrt[3]{2^4}=2\sqrt[3]{2}$ and $\left(\frac{R}{r}\right)^{n-\frac{4}{3}}-\frac{3}{2}\geq 2^{n-\frac{4}{3}}-\frac{3}{2}\overset{n\geq 2}{>}0.$

So the proof is complete. Equality holds iff ΔABC is equilateral.

1485.

In any acute triangle ABC, the following relationship holds :

$$(p-a)\sqrt{\cot A}+(p-b)\sqrt{\cot B}+(p-c)\sqrt{\cot C}>\frac{2}{3}p$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\geq b\geq c$ and then : $p-a\leq p-b\leq p-c$
and $\sqrt{\cot A}\leq\sqrt{\cot B}\leq\sqrt{\cot C}\therefore(p-a)\sqrt{\cot A}+(p-b)\sqrt{\cot B}+(p-c)\sqrt{\cot C}$

$$\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{cyc}(p-a)\right)\left(\sum_{cyc}\sqrt{\cot A}\right)\overset{?}{>}\frac{2}{3}p\Leftrightarrow\sum_{cyc}\sqrt{\cot A}\overset{?}{>}2\to(1)$$

Let $\sqrt{\cot A}=x,\sqrt{\cot B}=y,\sqrt{\cot C}=z\therefore(1)\Leftrightarrow\sum_{cyc}x>2\Leftrightarrow\left(\sum_{cyc}x\right)^2>4=$

$$4.\ \sqrt{\sum_{cyc} x^2y^2}\left(\because \sum_{cyc}\cot A\cot B = \sum_{cyc} x^2y^2 = 1\right) \Leftrightarrow \left(\sum_{cyc} x\right)^4 > 16\sum_{cyc} x^2y^2 \to (2)$$

Assigning $y+z=X, z+x=Y, x+y=Z \Rightarrow X+Y-Z=2z>0, Y+Z-X=2x$ >0 and $Z+X-Y=2y>0 \Rightarrow X+Y>Z, Y+Z>X, Z+X>Y \Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (i) \Rightarrow x = s-X, y = s-Y,$$

$$z = s-Z \text{ and such substitutions} \Rightarrow \sum_{cyc} xy = \sum_{cyc}(s-X)(s-Y)$$

$$\Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (ii) \text{ and } \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right)$$

$$\overset{\text{via (i) and (ii)}}{=} (4Rr+r^2)^2 - 2\left(\prod_{cyc}(s-X)\right).s = (4Rr+r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} x^2y^2 = r^2\left((4R+r)^2 - 2s^2\right) \to (iii) \therefore \text{via (i) and (iii)}, (2) \Leftrightarrow$$

$$s^4 > 16r^2\left((4R+r)^2 - 2s^2\right) \Leftrightarrow s^4 + 32r^2s^2 \overset{(*)}{>} 16r^2(4R+r)^2$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (16Rr+27r^2)s^2 \overset{\text{Gerretsen}}{\geq} (16Rr+27r^2)(16Rr-5r^2)$$

$$\overset{?}{>} 16r^2(4R+r)^2 \Leftrightarrow 76r(R-2r) + 148Rr + r^2 \overset{?}{>} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (*) \Rightarrow (2) \Rightarrow (1) \text{ is true}$$

$$\therefore (p-a)\sqrt{\cot A} + (p-b)\sqrt{\cot B} + (p-c)\sqrt{\cot C} > \frac{2}{3}p\ \forall \text{ acute } \Delta ABC \text{ (QED)}$$

1486.

In any ΔABC, the following relationship holds:

$$① \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4 + \left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4 + \left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4 \geq \frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2 - 512r^8}$$

$$② \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5 + \left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5 + \left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5 \geq \frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2 - 2^{11}r^{10}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

Proof of ①:

$$\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4 + \left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4 + \left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4$$

$$\geq \left(\frac{h_a}{w_b}+\frac{w_c}{h_a}\right)^4 + \left(\frac{h_b}{w_c}+\frac{w_a}{h_b}\right)^4 + \left(\frac{h_c}{w_a}+\frac{w_b}{h_c}\right)^4 \overset{A-G}{\geq} 16\left(\left(\frac{w_c}{w_b}\right)^2 + \left(\frac{w_a}{w_c}\right)^2 + \left(\frac{w_b}{w_a}\right)^2\right)$$

$$\geq 16\left(\left(\frac{w_c}{w_b}\right)\left(\frac{w_a}{w_c}\right)+\left(\frac{w_a}{w_c}\right)\left(\frac{w_b}{w_a}\right)+\left(\frac{w_b}{w_a}\right)\left(\frac{w_c}{w_b}\right)\right)=16\left(\frac{w_a}{w_b}+\frac{w_b}{w_c}+\frac{w_c}{w_a}\right)$$

$$=16\left(\frac{w_a^2}{w_aw_b}+\frac{w_b^2}{w_bw_c}+\frac{w_c^2}{w_cw_a}\right)\overset{\substack{\text{Bergstrom }+\, w_a\geq h_a \text{ and analogs}\\ \text{and}\\ w_a\leq\sqrt{s(s-a)}\text{ and analogs}}}{\geq}$$

$$16.\frac{\left(\sum_{cyc}\frac{2rs}{a}\right)^2}{\sum_{cyc}\left(\sqrt{s(s-a)}\sqrt{s(s-b)}\right)}\overset{\substack{\text{Bergstrom}|\\ \text{and}\\ \text{CBS}}}{\geq}\frac{16\left(\frac{2rs.9}{2s}\right)^2}{s.\sqrt{3s-2s}.\sqrt{3s-2s}}=\frac{16.81r^2}{s^2}\overset{\text{Mitrinovic}}{\geq}$$

$$\frac{2^6.81r^2}{27R^2}\overset{?}{\geq}\frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2-512r^8}\Leftrightarrow 3\left(3R^4-32r^4\right)^2\underset{(*)}{\overset{?}{\geq}}512r^8+64R^2r^6$$

$$\text{Now, }512r^8+64R^2r^6\overset{\text{Euler}}{\leq}128R^2r^6+64R^2r^6=3.64R^2r^6\overset{?}{\leq}3\left(3R^4-32r^4\right)^2$$

$$\Leftrightarrow 3R^4-32r^4\overset{?}{\geq}8Rr^3\Leftrightarrow 3R\left(R^3-8r^3\right)+16r^3(R-2r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\therefore\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4\geq\frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2-512r^8}$$

Proof of ② :

$$\text{Now, }\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3\overset{A-G}{\geq}$$

$$3\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)\overset{A-G}{\geq}3.8.\sqrt{\frac{m_a}{w_b}.\frac{w_c}{h_a}.\frac{m_b}{w_c}.\frac{w_a}{h_b}.\frac{m_c}{w_a}.\frac{w_b}{h_c}}\geq 3.8\overset{?}{\geq}$$

$$\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}\Leftrightarrow\left(9R^3-64r^3\right)^2\overset{?}{\geq}64r^6\Leftrightarrow 9R^3-64r^3\overset{?}{\geq}8r^3$$

$$\Leftrightarrow R^3\overset{?}{\geq}8r^3\to\text{true}\because R\overset{\text{Euler}}{\geq}2r\therefore\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3$$

$$\geq\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}\forall\Delta ABC\to(1)$$

$$\text{Via Chebyshev, }\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5\geq$$

$$\frac{1}{3}\left(\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^2+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^2+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^2\right)\left(\begin{array}{c}\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\\ \left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3\end{array}\right)$$

$$\overset{\text{via }(1)}{\geq}\frac{1}{3}\left(\left(\frac{h_a}{w_b}+\frac{w_c}{h_a}\right)^2+\left(\frac{h_b}{w_c}+\frac{w_a}{h_b}\right)^2+\left(\frac{h_c}{w_a}+\frac{w_b}{h_c}\right)^2\right).\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}\overset{A-G}{\geq}$$

$$\frac{4}{3}\left(\frac{w_c}{w_b}+\frac{w_a}{w_c}+\frac{w_b}{w_a}\right).\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}\overset{A-G}{\geq}\frac{4}{3}.3.\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}$$

$$\overset{?}{\geq}\frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2-2^{11}r^{10}}$$

$$\Leftrightarrow 3\left(81R^5-2560r^5\right)^2-2^{11}r^{10}\overset{?}{\geq}48r^4\left(9R^3-64r^3\right)^2-2^4.r^4.2^7.r^6$$

$$\Leftrightarrow 81R^5-2560r^5\overset{?}{\geq}4r^2(9R^3-64r^3)\Leftrightarrow 9R^5-4R^3r^2-256r^5\overset{?}{\geq}0$$

$$\Leftrightarrow R^3(R^2-4r^2)+8(R^5-32r^5)\overset{?}{\geq}0\to \text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\therefore\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5\geq\frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2-2^{11}r^{10}}$$

$$\therefore ①\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4\geq\frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2-512r^8}\ and$$

$$②\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5\geq\frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2-2^{11}r^{10}}$$

$\forall\,\Delta\,ABC$, equalities iff $\Delta\,ABC$ is equilateral (QED)

1487.

In any Δ ABC, the following relationship holds :

$$\left(\frac{m_a}{w_b}\right)^2+\left(\frac{w_b}{h_c}\right)^2+\left(\frac{h_c}{m_a}\right)^2\leq 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{h_b}\right)^2+\left(\frac{m_b}{h_c}\right)^2+\left(\frac{m_c}{h_a}\right)^2=\sum_{cyc}\frac{2b^2+2c^2+2a^2-3a^2}{4h_b^2}$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2\right)\left(\frac{1}{4r^2s^2}\right)\left(\sum_{cyc}a^2\right)-\frac{3}{4.4r^2s^2}\sum_{cyc}a^2b^2\overset{\substack{\text{Leibnitz}\\ and\\ \text{Gerretsen}}}{\leq}$$

$$\frac{81R^4}{4r^2.(27Rr+5r(R-2r))}-\frac{3abc(a+b+c)}{16r^2s^2}\overset{\text{Euler}}{\leq}\frac{81R^4}{4r^2.27Rr}-\frac{3.4Rrs.2s}{16r^2s^2}$$

$$=\frac{3R^3}{4r^3}-\frac{3R}{2r}=\frac{3R^3-6Rr^2}{4r^3}\overset{?}{\leq}9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6=\frac{9(9R^3-64r^3)^2-384r^6}{64r^6}$$

$$\Leftrightarrow 3(9R^3-64r^3)^2-128r^6\overset{?}{\geq}16r^3(3R^3-6Rr^2)$$

$$\Leftrightarrow 243t^6-3472t^3+32t+12160\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\left((t-2)(243t^4+972t^3+2916t^2+4304t+5552)+5024\right)\overset{?}{\geq}0\to\text{true}$$

$$\Rightarrow\left(\frac{m_a}{w_b}\right)^2+\left(\frac{w_b}{h_c}\right)^2+\left(\frac{h_c}{m_a}\right)^2\leq 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6$$

$\forall\,\Delta\,ABC$,'' ='' iff $\Delta\,ABC$ is equilateral (QED)

1488. In any $\Delta\, ABC$, the following relationship holds :

$$\frac{m_a+m_b}{w_b+w_c}+\frac{w_b+w_c}{h_c+h_a}+\frac{h_c+h_a}{m_a+m_b}\le 9.\left(\frac{7}{8}.\left(\frac{R}{r}\right)^3-6\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a+m_b}{w_b+w_c}+\frac{w_b+w_c}{h_c+h_a}+\frac{h_c+h_a}{m_a+m_b}\le\frac{m_a+m_b}{h_b+h_c}+\frac{m_b+m_c}{h_c+h_a}+\frac{m_c+m_a}{h_a+h_b}$$

$$=\sum_{cyc}\frac{m_b+m_c+m_a-m_a}{h_c+h_a}\le(4R+r)\sum_{cyc}\frac{1}{h_c+h_a}-\sum_{cyc}\frac{m_a}{h_c+h_a}$$

$$\le 2R(4R+r)\sum_{cyc}\frac{1}{b(c+a)}-\sum_{cyc}\frac{h_a}{h_c+h_a}$$

$$=2R(4R+r)\sum_{cyc}\frac{ca\left(b^2+\sum_{cyc}ab\right)}{abc(a+b)(b+c)(c+a)}-\sum_{cyc}\frac{bc}{ab+bc}\overset{Cesaro}{\le}$$

$$\frac{2R(4R+r)}{8.16R^2r^2s^2}\left(abc\sum_{cyc}a+\left(\sum_{cyc}ab\right)^2\right)-\sum_{cyc}\frac{c^2}{ac+c^2}\overset{Bergstrom}{\le}$$

$$\frac{8R(4R+r)}{8.16R^2r^2s^2}\sum_{cyc}a^2b^2-\frac{4s^2}{s^2+4Rr+r^2+2(s^2-4Rr-r^2)}\overset{Goldstone}{\le}$$

$$\frac{8R(4R+r).4R^2s^2}{8.16R^2r^2s^2}-\frac{4s^2}{3s^2-4Rr-r^2}=$$

$$\frac{R(4R+r)}{4r^2}-\frac{4s^2}{3s^2-4Rr-r^2}\overset{?}{\le}9.\left(\frac{7}{8}.\left(\frac{R}{r}\right)^3-6\right)$$

$$\Leftrightarrow(63R^3-24R^2r-6Rr^2-400r^3)s^2\underset{(*)}{\overset{?}{\ge}}$$

$$r(84R^4-11R^3r-16R^2r^2-578Rr^3-144r^4)$$

$\boxed{\text{Case 1}}$ $63R^3-24R^2r-6Rr^2-400r^3\ge 0$ and then :

$$(63R^3-24R^2r-6Rr^2-400r^3)s^2\overset{Gerretsen}{\ge}$$

$$(63R^3-24R^2r-6Rr^2-400r^3)(16Rr-5r^2)$$

$$\overset{?}{\ge}r(84R^4-11R^3r-16R^2r^2-578Rr^3-144r^4)$$

$$\Leftrightarrow 231t^4-172t^3+10t^2-1448t+536\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(231t^3+290t^2+456t+134(t-2)\right)\overset{?}{\ge}0$$

$$\rightarrow \text{true} \because t\overset{Euler}{\ge}2 \therefore (*) \text{ is true}$$

$\boxed{\text{Case 2}}$ $63R^3-24R^2r-6Rr^2-400r^3<0$ and then :

$$(63R^3-24R^2r-6Rr^2-400r^3)s^2$$

$$=-\left(-\left(63R^3-24R^2r-6Rr^2-400r^3\right)\right)s^2\overset{Gerretsen}{\geq}$$
$$-\left(-\left(63R^3-24R^2r-6Rr^2-400r^3\right)\right)\left(4R^2+4Rr+3r^2\right)$$
$$\overset{?}{\geq} r\left(84R^4-11R^3r-16R^2r^2-578Rr^3-144r^4\right)$$
$$\Leftrightarrow 63t^5+18t^4+20t^3-420t^2-260t-264\overset{?}{\geq}0$$
$$\Leftrightarrow (t-2)\left(63t^4+144t^3+308t^2+196t+132\right)\overset{?}{\geq}0\rightarrow \text{true} \because t\overset{Euler}{\geq}2$$
$\therefore$ $(*)$ is true $\therefore$ combining cases 1 and 2, $(*)$ is true $\forall\,\Delta\,ABC$
$$\therefore \frac{m_a+m_b}{w_b+w_c}+\frac{w_b+w_c}{h_c+h_a}+\frac{h_c+h_a}{m_a+m_b}\leq 9.\left(\frac{7}{8}.\left(\frac{R}{r}\right)^3-6\right)$$
$\forall\,\Delta\,ABC$,'' ='' iff $\Delta\,ABC$ is equilateral (QED)

1489. In acute ΔABC, H –orthocenter, $AH=d_a, BH=d_b, CH=d_c$. Prove that:

$$\sum_{cyc} tanA\cdot\frac{d_a}{h_a}=2\sum_{cyc} cotA$$

Proposed by Ertan Yildirim-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} cotA=\sum_{cyc}\frac{cosA}{sinA}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bcsinA}=\sum_{cyc}\frac{b^2+c^2-a^2}{4F}=$$
$$=\frac{1}{4F}\left(\sum_{cyc}b^2+\sum_{cyc}c^2-\sum_{cyc}a^2\right)=\frac{1}{4F}\sum_{cyc}a^2$$

$$\sum_{cyc} cotA=\frac{a^2+b^2+c^2}{4F}\quad (1)$$

Let be :

$$CC'\perp AB,\qquad cosA=\frac{AC'}{b}\Rightarrow AC'=bcosA$$

$$cos(\sphericalangle BAA')=\frac{AC'}{AH}\Rightarrow cos\left(\frac{\pi}{2}-B\right)=\frac{bcosA}{AH}$$

$$AH=\frac{2RsinBcosA}{sinB}=2RcosA=d_a$$

$$\sum_{cyc} tanA\cdot\frac{d_a}{h_a}=\sum_{cyc} tanA\cdot\frac{2RcosA}{\frac{2F}{a}}=\sum_{cyc}\frac{sinA}{cosA}\cdot\frac{a}{2F}\cdot 2RcosA=\sum_{cyc}\frac{aRsinA}{F}=$$

$$=\frac{1}{2F}\sum_{cyc} a\cdot 2RsinA=\frac{1}{2F}\sum_{cyc} a^2=2\cdot\frac{a^2+b^2+c^2}{4F}\overset{(1)}{\cong}2\sum_{cyc} cotA$$

1490. Let ΔDEF be the orthic triangle of acute $\Delta ABC, D\in(BC), E\in(CA),$ $F\in(AB), r_1, r_2, r_3$ –inradii of $\Delta AFE, \Delta BDF, \Delta CED$ respectively.

Prove that:

$$\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}=\frac{R}{r}\cdot\sum_{cyc} tanA$$

Proposed by Mehmet Şahin-Turkiye

Solution by Daniel Sitaru-Romania

$$AF=ACcosA=bcosA,\qquad AE=ABcosA=ccosA$$

$$EF^2=AF^2+AE^2-2AF\cdot AEcosA=b^2cos^2A+c^2cos^2A-2bccos^3A=$$

$$=(b^2+c^2-2bccosA)cos^2A=a^2cos^2A$$

$$EF=acosA$$

$$r_1=\frac{[AEF]}{\frac{AE+AF+EF}{2}}=\frac{\frac{1}{2}\cdot AE\cdot AF\cdot sinA}{\frac{acosA+bcosA+ccosA}{2}}=\frac{bcosA\cdot ccosA\cdot sinA}{(a+b+c)cosA}=$$

$$=\frac{bcsinA\cdot cosA}{2s}=\frac{4FcosA}{2s}=\frac{2rscosA}{s}=2rcosA$$

$$\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}=\frac{2RsinA}{2rcosA}+\frac{2RsinB}{2rcosB}+\frac{2RsinC}{2rcosC}=\frac{R}{r}\cdot\sum_{cyc} tanA$$

1491. In ΔABC the following relationship holds:

$$a^8+b^8+c^8\geq\frac{256}{3}\cdot F^4+\frac{1}{2}\sum_{cyc}(a^4-b^4)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

We know that

$$16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4) \qquad \textbf{(1)}$$

Since, $(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2 \geq 0$

i.e., $b^2c^2 + c^2a^2 + a^2b^2 \leq a^4 + b^4 + c^4$

So, we can say from (1): $16F^2 \leq b^2c^2 + c^2a^2 + a^2b^2$

$$\therefore F^4 \leq \frac{1}{256}(a^2b^2 + b^2c^2 + c^2a^2)^2 \qquad \textbf{(2)}$$

$$\frac{1}{2}\left[\sum(a^4 - b^4)^2\right] = \frac{1}{2}\left[2\left(\sum a^8\right) - 2\left(\sum a^4b^4\right)\right] = \left(\sum a^8\right) - \sum a^4b^4$$

We need to show: $a^8 + b^8 + c^8 \geq \frac{256}{3}F^4 + \frac{1}{2}\sum(a^4 - b^4)^2$

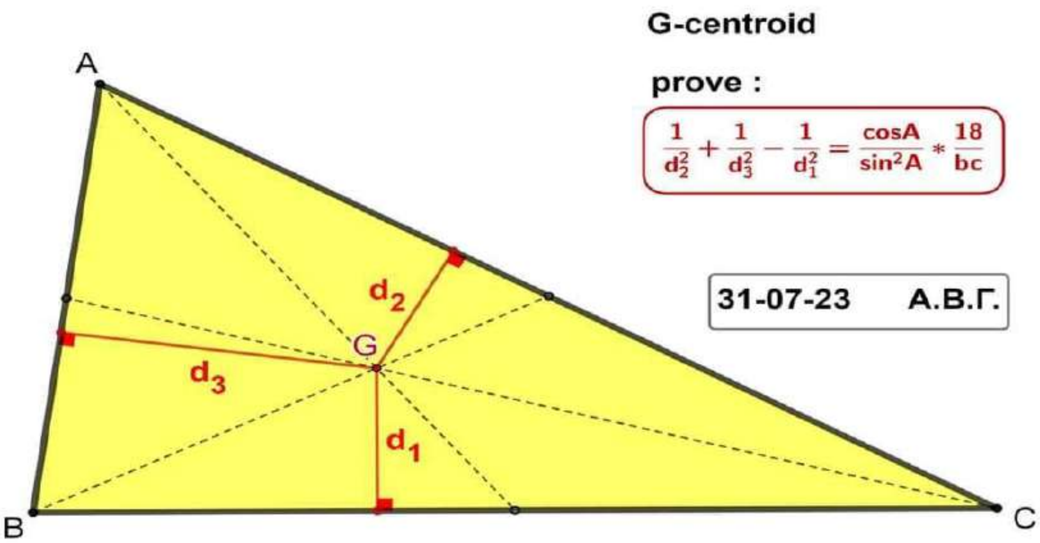
Or, $a^8 + b^8 + c^8 \geq \frac{256}{3}F^4 + (\sum a^8) - \sum a^4b^4$ Or $\sum a^4b^4 \geq \frac{256}{3}F^4$

Or, $\sum a^4b^4 \geq \frac{256}{3} \cdot \frac{1}{256}(b^2c^2 + c^2a^2 + a^2b^2)^2$

[Using relation (2)] Or $\sum a^4b^4 \geq \frac{1}{3}(b^2c^2 + c^2a^2 + a^2b^2)^2$

This is true by CBS inequality.

1492.



Proposed by Thanasis Gakopoulos-Greece

Solution by Daniel Sitaru-Romania

$$\frac{1}{d_2^2}+\frac{1}{d_3^2}-\frac{1}{d_1^2}=\frac{1}{\frac{4F^2}{9b^2}}+\frac{1}{\frac{4F^2}{9c^2}}-\frac{1}{\frac{4F^2}{9a^2}}=\frac{9(b^2+c^2-a^2)}{4F^2}=$$

$$=\frac{9(a^2+2bccosA-a^2)}{4F^2}=\frac{9bccosA}{2F^2}=\frac{9bccosA}{2\cdot\frac{1}{4}(bcsinA)^2}=\frac{18bccosA}{(bcsinA)^2}=\frac{18cosA}{bcsin^2A}$$

1493.

Let the quadrilateral ABCD circumscribe a circle of radius r and let A′B′C′D′ be the quadrilateral whose vertices are the points of contact of the sides of the quadrilateral ABCD with the circle. Prove that:

$$\sqrt[4]{r}\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)\ge\frac{64}{\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12}$$

Proposed by Radu Diaconu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $AA'=AD'=e, BB'=BA'=f, CC'=CB'=g$, $DD'=DC'=h$ and with reference to article ″Calculations concerning the Tangent Lengths and Tangency Chords of a Tangential Quadrilateral″ by Martin Josefsson, published in ″Forum Geometricum″, Volume 10 (2010) 119 − 130, we get:

$$r=\sqrt{\frac{\theta}{\xi}}\ (\xi=e+f+g+h \text{ and } \theta=efg+fgh+ghe+hef),$$

$$\sin\frac{A}{2}=\sqrt{\frac{\theta}{E}},\sin\frac{B}{2}=\sqrt{\frac{\theta}{F}},\sin\frac{C}{2}=\sqrt{\frac{\theta}{G}},\sin\frac{D}{2}=\sqrt{\frac{\theta}{H}},\text{where}$$

$$E=(e+f)(e+g)(e+h), F=(f+e)(f+g)(f+h),$$
$$G=(g+e)(g+f)(g+h) \text{ and } H=(h+e)(h+f)(h+g)$$

$$\text{and using }\cos\frac{A}{2}=\sqrt{1-\sin^2\frac{A}{2}}\text{ and analogs, we get: }\cos\frac{A}{2}=e.\sqrt{\frac{\xi}{E}},$$

$$\cos\frac{B}{2}=f.\sqrt{\frac{\xi}{F}},\cos\frac{C}{2}=g.\sqrt{\frac{\xi}{G}}\therefore\sin\frac{A+B}{2}=\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}$$

$$=\sqrt{\frac{\theta}{E}}.f.\sqrt{\frac{\xi}{F}}+e.\sqrt{\frac{\xi}{E}}.\sqrt{\frac{\theta}{F}}\Rightarrow \sin\frac{A+B}{2}=\sqrt{\frac{\theta\xi}{EF}}(e+f)\text{ and analogously},$$

$$\sin\frac{B+C}{2}=\sqrt{\frac{\theta\xi}{FG}}(f+g)\text{ and }\sin\frac{C+A}{2}=\sqrt{\frac{\theta\xi}{GE}}(g+e)\therefore\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}$$

$$=\frac{\left(\sqrt{\frac{\theta\xi}{EF}}(e+f)\right)\left(\sqrt{\frac{\theta\xi}{FG}}(f+g)\right)\left(\sqrt{\frac{\theta\xi}{GE}}(g+e)\right)}{\left(\sqrt{\frac{\theta}{E}}\right)\left(\sqrt{\frac{\theta}{F}}\right)\left(\sqrt{\frac{\theta}{G}}\right)\left(\sqrt{\frac{\theta}{H}}\right)}$$

$$=\frac{\xi.\sqrt{\frac{\xi}{\theta}}.\sqrt{EFGH}.(e+f)(f+g)(g+e)}{EFG}$$

$$=\frac{\left(\frac{\sum_{cyc}e}{r}\right)(e+f)(f+g)(g+e).\sqrt{(e+f)^2(e+g)^2(e+h)^2(f+g)^2(f+h)^2(h+e)^2}}{(e+f)^2(e+g)^2(f+g)^2(h+e)(h+f)(h+g)}$$

$$=\frac{\sum_{cyc}e}{r}\Rightarrow\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12=\frac{\sum_{cyc}e}{r}+12$$

$$\therefore\left(\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12\right).\sqrt[4]{r}.\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)$$

$$=\left(\frac{\sum_{cyc}e}{r}+12\right).\sqrt[4]{r}.\sum_{cyc}\frac{1}{\sqrt[4]{e}}\overset{\text{Jensen}}{\geq}\left(\frac{\sum_{cyc}e}{r}+12\right).\sqrt[4]{r}.\frac{4}{\sqrt[4]{\frac{\sum_{cyc}e}{4}}}$$

$$\left(\because f(x)=\frac{1}{\sqrt[4]{x}}\text{ is convex as }f''(x)=\frac{5}{16x^{\frac{9}{4}}}>0\right)=\frac{4\left(\frac{\sum_{cyc}e}{r}+12\right)}{\sqrt[4]{\frac{\sum_{cyc}e}{4r}}}=\frac{4(4t^4+12)}{t}$$

$$\left(t=\sqrt[4]{\frac{\sum_{cyc}e}{4r}}\right)\overset{?}{\geq}64\Leftrightarrow t^4-4t+3\overset{?}{\geq}0\Leftrightarrow(t^2+2t+3)(t-1)^2\overset{?}{\geq}0\to\text{true}$$

$$\therefore\left(\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12\right).\sqrt[4]{r}.\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)\geq 64$$

$$\Rightarrow\sqrt[4]{r}\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)\geq\frac{64}{\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12}\text{ (QED)}$$

1494. In any ΔABC with $k^3 = h_a h_b h_c$ and

$t^3 = (abc)^2$, the following relationship holds

$$\frac{r_a^2}{k+a^2}+\frac{r_b^2}{k+b^2}+\frac{r_c^2}{k+c^2} \ge \frac{108r^3}{t+6R^3}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^2 r_a = rs\sum_{cyc}\frac{a^2}{s-a} = rs\sum_{cyc}\frac{(a-s+s)^2}{s-a}$$

$$= rs\left(\sum_{cyc}\frac{(s-a)^2}{s-a} - 2s\sum_{cyc}\frac{s-a}{s-a} + \frac{s^2}{r^2 s}\sum_{cyc}(s-b)(s-c)\right)$$

$$= rs\left(s - 6s + \frac{s(4R+r)}{r}\right) \Rightarrow \sum_{cyc} a^2 r_a = 4(R-r)s^2 \to (1)$$

$$\text{Now, } \frac{\sqrt[3]{a^2b^2c^2}.(4R+r)^3}{(4R+r).324r^3.\sqrt[3]{h_a h_b h_c}} \overset{\text{Euler}}{\ge} \frac{\sqrt[3]{16R^2r^2s^2}.(9r)^2}{324r^3.\sqrt[3]{\frac{2r^2s^2}{R}}} = \frac{R}{2r} \overset{\text{Euler}}{\ge} 1$$

$$\Rightarrow t(4R+r)^3 \ge 324r^3 k(4R+r) \to (2)$$

$$\text{Also, } \frac{R^3(4R+r)^3}{216(R-r)r^3s^2} \overset{\text{Mitrinovic}}{\ge} \frac{4R^3(4R+r)^3}{216(R-r)r^3.27R^2} \overset{?}{\ge} 1$$

$$\Leftrightarrow R(4R+r)^3 \overset{?}{\ge} 1458(R-r)r^3$$

$$\Leftrightarrow 64t^4 + 48t^3 + 12t^2 - 1457t + 1458 \overset{?}{\ge} 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(64t^2+304t+972)+1215\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow 6R^3(4R+r)^3 \ge 108.12(R-r)r^3s^2 \to (3)$$

$$\text{So, } \frac{r_a^2}{k+a^2}+\frac{r_b^2}{k+b^2}+\frac{r_c^2}{k+c^2} = \frac{r_a^3}{kr_a+a^2r_a}+\frac{r_b^3}{kr_b+b^2r_b}+\frac{r_c^3}{kr_c+c^2r_c}$$

$$\overset{\substack{\text{Holder}\\ \text{and}\\ \text{via (1)}}}{\ge} \frac{(4R+r)^3}{3k(4R+r)+12(R-r)s^2} \overset{?}{\ge} \frac{108r^3}{t+6R^3}$$

$$\Leftrightarrow t(4R+r)^3 + 6R^3(4R+r)^3 \overset{?}{\ge} 324r^3k(4R+r) + 108.12(R-r)r^3s^2$$

$$\to \text{true via } (2)+(3) \therefore \frac{r_a^2}{k+a^2}+\frac{r_b^2}{k+b^2}+\frac{r_c^2}{k+c^2} \ge \frac{108r^3}{t+6R^3}$$

$\forall \Delta ABC, '' ='' $ iff ΔABC is equilateral (QED)

1495. If H −orthocenter in acute $\Delta ABC, AD, BE, CF$ −altitudes, $HD = x$, $HE = y, HF = z$ then:

$$\frac{x}{bc}+\frac{y}{ca}+\frac{z}{ab}\le\frac{1}{4r}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$cosB=\frac{BD}{AB}\Rightarrow BD=ccosB$$

$$tan(\sphericalangle HBD)=\frac{HD}{BD}\Rightarrow HD=BDtan\left(\frac{\pi}{2}-C\right)=ccosBtan\left(\frac{\pi}{2}-C\right)=$$

$$=ccosBcotC=ccosB\cdot\frac{cosC}{sinC}=2RsinCcosB\cdot\frac{cosC}{sinC}=2RcosBcosC$$

$$\frac{x}{bc}+\frac{y}{ca}+\frac{z}{ab}=\frac{2RcosBcosC}{bc}+\frac{2RcosCcosA}{ca}+\frac{2RcosAcosB}{ab}=$$

$$=\frac{2RcosBcosC}{2RsinB\cdot 2RsinC}+\frac{2RcosCcosA}{2RsinC\cdot 2RsinA}+\frac{2RcosAcosB}{2RsinA\cdot 2RsinB}=$$

$$=\frac{cotBcotC}{2R}+\frac{cotCcotA}{2R}+\frac{cotAcotB}{2R}=$$

$$=\frac{1}{2R}\sum_{cyc}cotBcotC=\frac{1}{2R}\cdot 1\overset{EULER}{\le}\frac{1}{2\cdot 2r}=\frac{1}{4r}$$

Equality holds for: $a=b=c$.

1496. In any Δ ABC, the following relationship holds :

$$p^2+\frac{\lambda}{\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}}\ge(27+3\lambda)r^2, 0\le\lambda\le\frac{9}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

For own convenience, $p\equiv s$

$$\text{Firstly}, \frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\ge\frac{1}{3}\left(\sum_{cyc}\frac{1}{r_a}\right)^2=\frac{1}{3r^2}\Rightarrow\sum_{cyc}\frac{1}{r_a^2}\ge\frac{1}{3r^2}$$

$$\Rightarrow 3r^2-\frac{1}{\sum_{cyc}\frac{1}{r_a^2}}\ge 0\to(1)$$

$$s^2+\frac{\lambda}{\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}}\ge(27+3\lambda)r^2\Leftrightarrow s^2-27r^2\ge\lambda\left(3r^2-\frac{1}{\sum_{cyc}\frac{1}{r_a^2}}\right)$$

to prove which it suffices to prove :

$$s^2-27r^2\ge\frac{9}{2}\left(3r^2-\frac{1}{\sum_{cyc}\frac{1}{r_a^2}}\right)\left(\text{via } (1) \text{ and } \because 0\le\lambda\le\frac{9}{2}\right)$$

$$\Leftrightarrow s^2-27r^2\ge\frac{9}{2}\left(3r^2-\frac{r^2s^4}{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}\right)$$

$$=\frac{9}{2}\left(3r^2-\frac{r^2s^4}{s^4-2rs^2(4R+r)}\right)\Leftrightarrow s^2-27r^2\ge\frac{9r^2}{2}\left(\frac{2s^2-24Rr-6r^2}{s^2-8Rr-2r^2}\right)$$

$$\Leftrightarrow(s^2-8Rr-2r^2)(s^2-27r^2)\overset{(*)}{\ge}9r^2(s^2-12Rr-3r^2)$$

$$s^2-27r^2\overset{?}{\ge}s^2-12Rr-3r^2\Leftrightarrow 12Rr\overset{?}{\ge}24r^2\to\text{true via Euler}$$

$$\therefore s^2-27r^2\ge s^2-12Rr-3r^2\to(2)$$

$$\text{Also, } s^2-8Rr-2r^2\overset{?}{\ge}9r^2\Leftrightarrow s^2-16Rr+5r^2+8r(R-2r)\overset{?}{\ge}0\to\text{true}$$

$$\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0 \text{ and } 8r(R-2r)\overset{\text{Euler}}{\ge}0\therefore s^2-8Rr-2r^2\ge 9r^2\to(3)$$

$$\therefore(2)\bullet(3)\Rightarrow(*)\text{ is true }\therefore$$

$$p^2+\frac{\lambda}{\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}}\ge(27+3\lambda)r^2\ \forall\,\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1497. In any ΔABC, the following relationship holds :

$$\frac{8R}{r}\le\sum_{cyc}\frac{(b+c)^2}{r_a^2}\le 16\left(\frac{R^2}{r^2}-3\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\ge b\ge c$ and then :

$$(b+c)^2\le(c+a)^2\le(a+b)^2 \text{ and } \frac{1}{r_a^2}\le\frac{1}{r_b^2}\le\frac{1}{r_c^2}\therefore\text{ via Chebyshev,}$$

$$\sum_{cyc}\frac{(b+c)^2}{r_a^2}\ge\left(\sum_{cyc}(b+c)^2\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right)$$

$$\overset{A-G}{\ge}4\left(\sum_{cyc}ab\right).\frac{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}{r^2s^4}$$

$$=4(s^2+4Rr+r^2).\frac{s^4-2rs^2(4R+r)}{r^2s^4}\overset{?}{\ge}\frac{8R}{r}$$

$$\Leftrightarrow s^4 - (10Rr + r^2)s^2 - 2r^2(4R+r)^2 \underset{(*)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (6Rr - 6r^2)s^2 - 2r^2(4R+r)^2 \overset{\text{Gerretsen}}{\geq}$$

$$(6Rr - 6r^2)(16Rr - 5r^2) - 2r^2(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow 32R^2 - 71Rr + 14r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(32R-7r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{(b+c)^2}{r_a^2} \geq \frac{8R}{r}$$

$$\text{Again, } \sum_{cyc}\left((b+c)^2(s-a)^2\right) = \sum_{cyc}\left((s+s-a)^2(s-a)^2\right)$$

$$= \sum_{cyc}\left(\left(s^2 + (s-a)^2 + 2s(s-a)\right)(s-a)^2\right)$$

$$= s^2\sum_{cyc}(s^2 - 2sa + a^2) + 2s\sum_{cyc}(s^3 - 3s^2a + 3sa^2 - a^3)$$

$$+\sum_{cyc}(s^4 - 4s^3a + 6s^2a^2 - 4sa^3 + a^4)$$

$$= s^2.3s^2 - 2s^3.2s + s^2\sum_{cyc}a^2 + 2s.3s^3 - 6s^3.2s + 6s^2\sum_{cyc}a^2 - 2s\sum_{cyc}a^3 + 3s^4$$

$$-4s^3.2s + 6s^2\sum_{cyc}a^2 - 4s\sum_{cyc}a^3 + 2\sum_{cyc}a^2b^2 - 16r^2s^2$$

$$= -12s^4 + 26s^2(s^2 - 4Rr - r^2) - 12s^2(s^2 - 6Rr - 3r^2)$$

$$+2\left((s^2 + 4Rr + r^2)^2 - 16Rrs^2\right) - 16r^2s^2$$

$$= 2\left(2s^4 - (24Rr + r^2)s^2 + r^2(4R+r)^2\right)$$

$$\Rightarrow \sum_{cyc}\frac{(b+c)^2}{r_a^2} = \frac{2\left(2s^4 - (24Rr + r^2)s^2 + r^2(4R+r)^2\right)}{r^2s^2} \overset{?}{\leq} 16\left(\frac{R^2}{r^2} - 3\right)$$

$$\Leftrightarrow 2s^4 + r^2(4R+r)^2 \underset{(**)}{\overset{?}{\leq}} (8R^2 + 24Rr - 23r^2)s^2$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\leq} (8R^2 + 8Rr + 6r^2)s^2 + r^2(4R+r)^2 \overset{?}{\leq}$$

$$(8R^2 + 24Rr - 23r^2)s^2 \Leftrightarrow (16Rr - 29r^2)s^2 \underset{(***)}{\overset{?}{\geq}} r^2(4R+r)^2$$

$$\text{Again, LHS of } (***) \overset{\text{Gerretsen}}{\geq} (16Rr - 29r^2)(16Rr - 5r^2) \overset{?}{\geq} r^2(4R+r)^2$$

$$\Leftrightarrow 240R^2 - 552Rr + 144r^2 \overset{?}{\geq} 0 \Leftrightarrow 24(R-2r)(10R-3r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (***) \Rightarrow (**) \text{ is true} \therefore \sum_{cyc}\frac{(b+c)^2}{r_a^2} \leq 16\left(\frac{R^2}{r^2} - 3\right) \therefore \frac{8R}{r} \leq \sum_{cyc}\frac{(b+c)^2}{r_a^2}$$

$$\leq 16\left(\frac{R^2}{r^2} - 3\right) \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1498. In any Δ ABC, the following relationship holds :

$$\frac{9R}{2r} \le \sum_{cyc} \frac{r_a}{r_b} \sum_{cyc} \frac{r_b}{r_a} \le \frac{8R^2}{r^2} - 23$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2} \text{ and analogs}$$

$$\text{Now, } \sum_{cyc} \frac{r_a}{r_b} \sum_{cyc} \frac{r_b}{r_a} = 3 + \sum_{cyc} \frac{r_b r_c}{r_a^2} + \sum_{cyc} \frac{r_a^2}{r_b r_c}$$

$$= 3 + \frac{1}{r^2 s}\sum_{cyc}(s-a)^3 + \sum_{cyc}\frac{(s-b)(s-c)}{(s-a)^2} \overset{A-G}{\le} 3 +$$

$$\frac{1}{r^2 s}\left(\begin{array}{c}\left(\sum_{cyc}(s-a)\right)^3 \\ -3\big((s-a)+(s-b)\big)\big((s-b)+(s-c)\big)\big((s-c)+(s-a)\big)\end{array}\right)$$

$$+\frac{1}{4}\sum_{cyc}\frac{(a-s+s)^2}{(s-a)^2} = 3 + \frac{1}{r^2 s}(s^3 - 3.4Rrs) + \frac{1}{4}\sum_{cyc}\frac{(s-a)^2 - 2s(s-a) + s^2}{(s-a)^2}$$

$$= 3 + \frac{s^2 - 12Rr}{r^2} + \frac{1}{4}\left(3 - \frac{2}{r}\sum_{cyc} r_a + \frac{1}{r^2}\sum_{cyc} r_a^2\right)$$

$$= 3 + \frac{s^2 - 12Rr}{r^2} + \frac{1}{4}\left(3 - \frac{2(4R+r)}{r} + \frac{(4R+r)^2 - 2s^2}{r^2}\right)$$

$$= 3 + \frac{s^2 - 12Rr}{r^2} + \frac{8R^2 + r^2 - s^2}{2r^2} \overset{?}{\le} \frac{8R^2}{r^2} - 23 \Leftrightarrow s^2 \overset{?}{\le} 8R^2 + 24Rr - 53r^2$$

$$\text{and } \because s^2 \overset{\text{Gerretsen}}{\le} 4R^2 + 4Rr + 3r^2 \therefore \text{it suffices to prove : } 4R^2 + 4Rr + 3r^2$$

$$\overset{?}{\le} 8R^2 + 24Rr - 53r^2 \Leftrightarrow R^2 + 5Rr - 14r^2 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(R+7r) \overset{?}{\ge} 0 \to \text{true}$$

$$\because R \overset{\text{Euler}}{\ge} 2r \therefore \boxed{\sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} \le \sum_{cyc} r_b r_c(r_b + r_c) \le \frac{8R^2}{r^2} - 23}$$

$$\text{Again, } \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} = 3 + \sum_{cyc}\frac{r_b r_c}{r_a^2} + \sum_{cyc}\frac{r_a^2}{r_b r_c} = 3$$

$$+\frac{1}{r^2s^4}\left(\left(\sum_{cyc} r_b r_c\right)^3 - 3(r_b r_c + r_c r_a)(r_c r_a + r_a r_b)(r_a r_b + r_b r_c)\right)$$

$$+\frac{1}{r^2 s}\left(\begin{array}{c}\left(\sum_{cyc}(s-a)\right)^3 \\ -3((s-a)+(s-b))((s-b)+(s-c))((s-c)+(s-a))\end{array}\right) \overset{via\ (i)}{=}$$

$$3+\frac{1}{r^2s^4}\left(s^4 - 3rs^2.64R^3.\frac{s^2}{16R^2}\right)+\frac{1}{r^2s}(s^3-3.4Rrs) \overset{?}{\geq} \frac{9R}{2r}$$

$$\Leftrightarrow \underbrace{2s^4-(57Rr-6r^2)s^2+2r(4R+r)^3 \overset{?}{\geq} 0}_{(*)} \ and \because 2(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \geq 2(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow 128R^3 - 416R^2r + 344Rr^2 - 48r^3 + 7(R-2r)s^2 \overset{(**)}{\geq} 0$$

$$\text{Again, LHS of } (**) \overset{Gerretsen}{\geq} 128R^3 - 416R^2r + 344Rr^2 - 48r^3$$

$$+7(R-2r)(16Rr-5r^2) \overset{?}{\geq} 0 \Leftrightarrow 128t^3 - 304t^2 + 85t + 22 \overset{?}{\geq} 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)((t-2)(128t+208)+405) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \boxed{\frac{9R}{2r} \leq \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a}} \text{ and hence,}$$

$$\frac{9R}{2r} \leq \sum_{cyc}\frac{r_a}{r_b}\sum_{cyc}\frac{r_b}{r_a} \leq \frac{8R^2}{r^2} - 23\ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1499. Prove that in *all* triangle ABC with *usual* notations, the following relationship holds

$$\frac{9}{\sum_{cyc} m_a} \leq \sum_{cyc}\frac{2}{m_a+m_b} < \frac{10}{\sum_{cyc} m_a}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{2}{a+b} < \frac{10}{\sum_{cyc} a} \Leftrightarrow \frac{1}{(a+b)(b+c)(c+a)}.\sum_{cyc}\left(a^2+\sum_{cyc} ab\right) < \frac{5}{2s} \Leftrightarrow$$

$$\frac{1}{2s(s^2+2Rr+r^2)}\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab + \sum_{cyc} ab\right) < \frac{5}{2s} \Leftrightarrow \frac{4s^2+s^2+4Rr+r^2}{s^2+2Rr+r^2} < 5$$

$$\Leftrightarrow 10Rr + 5r^2 > 4Rr + r^2 \Leftrightarrow 6Rr + 4r^2 > 0 \to \text{true} \therefore \sum_{cyc} \frac{2}{a+b} < \frac{10}{\sum_{cyc} a}$$

and implementing it on a triangle with sides m_a, m_b, m_c, we arrive at :

$$\sum_{cyc} \frac{2}{m_a + m_b} < \frac{10}{\sum_{cyc} m_a} \text{ and also, } \sum_{cyc} \frac{2}{m_a + m_b} \overset{\text{Bergstrom}}{\geq} \frac{18}{2\sum_{cyc} m_a}$$

$$\therefore \frac{9}{\sum_{cyc} m_a} \leq \sum_{cyc} \frac{2}{m_a + m_b} \text{ (QED)}$$

1500. Prove the following inequalities : (i) If $x, y, z > 0$, then :

$$\sum_{cyc} \frac{1}{x+y} \leq \sum_{cyc} \frac{4x}{3y^2 + 2yz + 3z^2} \leq \frac{\sum_{cyc} x^2}{2xyz} \text{ and (ii) In all triangle ABC}$$

with usual notations, the following relationship holds :

$$\frac{5s^2 + r^2 + 4Rr}{8s(s^2 + 2Rr + r^2)} \leq \sum_{cyc} \frac{a}{3b^2 + 2bc + 3c^2} \leq \frac{s^2 - r^2 - 4Rr}{16sRr}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c$

$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and, } \sum_{cyc} xy = \sum_{cyc} (s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2$$

$$\text{Firstly, } \sum_{cyc} \frac{4x}{3y^2 + 2yz + 3z^2} \overset{A-G}{\leq} \sum_{cyc} \frac{4x}{8yz} = \frac{\sum_{cyc} x^2}{2xyz}$$

$$\text{Again, } \sum_{cyc} \frac{4x}{3y^2 + 2yz + 3z^2} = \sum_{cyc} \frac{4x^2}{3xy^2 + 2xyz + 3xz^2} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{4\left(\sum_{cyc} x\right)^2}{3\sum_{cyc}\left(xy\left(\sum_{cyc} x - z\right)\right) + 6xyz} = \frac{4\left(\sum_{cyc} x\right)^2}{3\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 3xyz} \overset{\text{via } (\bullet), (\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=}$$

$$\frac{4s^2}{3s(4Rr + r^2) - 3r^2 s} \overset{?}{\geq} \sum_{cyc} \frac{1}{x+y} = \sum_{cyc} \frac{1}{a} = \frac{s^2 + 4Rr + r^2}{4Rrs}$$

$$\Leftrightarrow 4R(s^2 - 12Rr - 3r^2) \overset{?}{\geq} 0 \Leftrightarrow s^2 - 16Rr + 5r^2 + 4r(R - 2r) \overset{?}{\geq} 0 \to \text{true}$$

$$\because s^2 - 16Rr + 5r^2 \overset{\text{Gerretsen}}{\geq} 0 \text{ and } 4r(R - 2r) \overset{\text{Euler}}{\geq} 0$$

$$\therefore \sum_{cyc} \frac{4x}{3y^2+2yz+3z^2} \ge \sum_{cyc} \frac{1}{x+y} \text{ and hence,}$$

$$\boxed{\sum_{cyc} \frac{1}{x+y} \le \sum_{cyc} \frac{4x}{3y^2+2yz+3z^2} \le \frac{\sum_{cyc} x^2}{2xyz}} \quad '' = '' \text{ iff } x = y = z$$

and implementing this with $x \equiv a, y \equiv b, z \equiv c$ ($a, b, c \to$ sides of Δ ABC),

$$\text{we get : } \sum_{cyc} \frac{4a}{3b^2+2bc+3c^2} \le \frac{\sum_{cyc} a^2}{2abc} = \frac{s^2-r^2-4Rr}{4sRr}$$

$$\Rightarrow \sum_{cyc} \frac{a}{3b^2+2bc+3c^2} \le \frac{s^2-r^2-4Rr}{16sRr} \text{ and also, } \sum_{cyc} \frac{4a}{3b^2+2bc+3c^2} \ge \sum_{cyc} \frac{1}{b+c}$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \sum_{cyc} \left(a^2 + \sum_{cyc} ab\right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \left(\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab\right) + \sum_{cyc} ab\right) = \frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}$$

$$= \frac{5s^2+r^2+4Rr}{2s(s^2+2Rr+r^2)} \Rightarrow \sum_{cyc} \frac{a}{3b^2+2bc+3c^2} \ge \frac{5s^2+r^2+4Rr}{8s(s^2+2Rr+r^2)}$$

$$\therefore \boxed{\frac{5s^2+r^2+4Rr}{8s(s^2+2Rr+r^2)} \le \sum_{cyc} \frac{a}{3b^2+2bc+3c^2} \le \frac{s^2-r^2-4Rr}{16sRr}}$$

'' ='' iff Δ ABC is equilateral (QED)

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru