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SP.526 If $a,b,c,\lambda>0, a+b+c=\lambda$ then:

$$\sum\sqrt{\frac{bc}{a}+\lambda}\geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We obtain:

$$LHS=\sum\sqrt{\frac{bc}{a}+\lambda}=\sum\sqrt{\frac{bc}{a}+(a+b+c)}=\sum\sqrt{\frac{(a+b)(a+c)}{a}}\overset{(1)}{\geq}$$

$$\geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)=RHS,$$

where (1) $\Leftrightarrow \sum\sqrt{\frac{(a+b)(a+c)}{a}}\geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\Leftrightarrow$

$\Leftrightarrow \sum\sqrt{bc(a+b)(a+c)}\geq 2\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$, which follows from:

$$\sqrt{bc(a+b)(a+c)}\geq\sqrt{abc}\left(\sqrt{b}+\sqrt{c}\right)\Leftrightarrow\sqrt{(a+b)(a+c)}\geq\sqrt{a}\left(\sqrt{b}+\sqrt{c}\right)\Leftrightarrow$$

$$\Leftrightarrow (a+b)(a+c)\geq a\left(\sqrt{b}+\sqrt{c}\right)^2\Leftrightarrow\left(a-\sqrt{bc}\right)^2\geq 0.$$

Equality holds if and only if $a=b=c=\frac{\lambda}{3}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sqrt{\frac{bc}{a}+\lambda}=\sqrt{\frac{bc}{a}+a+b+c}=\sqrt{\left(\frac{b}{a}+1\right)(a+c)}\geq\sqrt{b}+\sqrt{c}.$$

Similarly, we have

$$\sqrt{\frac{ca}{b}+\lambda}\geq\sqrt{c}+\sqrt{a}\ \text{ and }\ \sqrt{\frac{ab}{c}+\lambda}\geq\sqrt{a}+\sqrt{b}.$$

Adding these inequalities yields the desired result.

Equality holds iff $a=b=c=\dfrac{\lambda}{3}$.