SP.527 If $x, y, z \geq 0$ with $x + y + z = 1$ and $0 \leq \lambda \leq \frac{9}{4}$,

$$\text{then}: xy + yz + zx - \lambda xyz \leq \frac{9-\lambda}{27}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

The inequality can be written equivalently. Denoting $yz = t$, the inequality can be written:

$$xy + t + zx - 2\lambda xt \leq \frac{9-\lambda}{27} \Leftrightarrow 27t(\lambda x - 1) - 27x(y+z) + 9 - \lambda \geq 0 \Leftrightarrow$$

$$27t(\lambda x - 1) - 27x(1-x) + 9 - \lambda \geq 0 \Leftrightarrow 27t(\lambda x - 1) + 27x(x-1) + 9 - \lambda \geq 0.$$

We use Lemma.

If the function $f: \mathbb{R} \to \mathbb{R}, f(x) = ax + b$, meets the conditions

$f(\alpha) \geq 0$ and $f(\beta) \geq 0$, where $a, b, \alpha, \beta \in \mathbb{R}$, then $f(t) \geq 0, \forall t \in [\alpha, \beta]$.

Let's get back to the main problem.

Denoting $yz = t$ we have $t = yz \overset{AGM}{\leq} \left(\frac{y+z}{2}\right)^2 \overset{x+y+z=1}{=} \left(\frac{1-x}{2}\right)^2 = t_0$.

$f(t) = 27t(\lambda x - 1) + 27x(x-1) + 9 - \lambda \geq 0$, where $0 \leq t \leq t_0 = \frac{(1-x)^2}{4}$.

We have:

$f(0) = 27x(x-1) + 9 - \lambda = 27x^2 - 27x + 9 - \lambda \geq 0$, because $\Delta = 27(4\lambda - 9) \leq 0$.

$$f(t_0) = f\left(\frac{(1-x)^2}{4}\right) = \frac{1}{4}(27\lambda x^3 + (81 - 54\lambda)x^2 + (27\lambda - 54)x + 1) =$$

$$= \frac{1}{4}(3x-1)^2(3\lambda x + 9 - 4\lambda) \geq 0$$

Using Lemma for $f(t) = 27t(\lambda x - 1) + 27x(x-1) + 9 - \lambda \geq 0$ and $\alpha = 0, \beta = \frac{(1-x)^2}{4}$,

we deduce the conclusion.

Equality holds if and only if $x = y = z = \frac{1}{3}$, for $\lambda < \frac{9}{4}$, and for $\lambda = \frac{9}{4}$ equality for

$$(x, y, z) \in \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right)\right\}.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Write the given inequality as follows

$$\lambda\left(\frac{1}{27} - xyz\right) \leq \frac{1}{3} - (xy + yz + zx).$$

By AM − GM inequality, we have $3\sqrt[3]{xyz} \le x + y + z = 1$, then $xyz \le \frac{1}{27}$.

So it suffices to prove that

$$\frac{9}{4}\left(\frac{1}{27} - xyz\right) \le \frac{1}{3} - (xy + yz + zx)$$
$$\Leftrightarrow 4(xy + yz + zx) \le 1 + 9xyz$$
$$\Leftrightarrow 4(xy + yz + zx)(x + y + z) \le (x + y + z)^3 + 9xyz$$
$$\Leftrightarrow xy(x + y) + yz(y + z) + zx(z + x) \le x^3 + y^3 + z^3 + 3xyz,$$

which is Schur's inequality.

So the proof is complete. Equality holds iff $x = y = z = \frac{1}{3}$.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$xy + yz + zx - \lambda xyz - \frac{9-\lambda}{27} \overset{x+y+z=1}{=}$$
$$\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - \lambda xyz - \left(\frac{9-\lambda}{27}\right)\left(\sum_{cyc} x\right)^3$$
$$= \lambda\left(\frac{\left(\sum_{cyc} x\right)^3}{27} - xyz\right) - \left(\sum_{cyc} x\right)\left(\frac{\left(\sum_{cyc} x\right)^2}{3} - \sum_{cyc} xy\right)$$
$$= \frac{\lambda}{27}\cdot\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) - \frac{\left(\sum_{cyc} x\right)}{3}\cdot\left(\left(\sum_{cyc} x\right)^2 - 3\sum_{cyc} xy\right)$$
$$\overset{0 \le \lambda \le \frac{9}{4}}{\le} \frac{\frac{9}{4}}{27}\cdot\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) - \frac{\left(\sum_{cyc} x\right)}{3}\cdot\left(\left(\sum_{cyc} x\right)^2 - 3\sum_{cyc} xy\right) \overset{?}{\le} 0$$
$$\Leftrightarrow 4\left(\sum_{cyc} x\right)\left(\left(\sum_{cyc} x\right)^2 - 3\sum_{cyc} xy\right) - \left(\sum_{cyc} x\right)^3 + 27xyz \overset{?}{\ge} 0$$
$$\Leftrightarrow \sum_{cyc} x^3 + 3xyz \overset{?}{\ge} \sum_{cyc} x^2y + \sum_{cyc} xy^2 \rightarrow \text{true via Schur} \therefore xy + yz + zx - \lambda xyz$$
$$\le \frac{9-\lambda}{27}\ \forall x, y, z \ge 0 \text{ with } x + y + z = 1 \text{ and } 0 \le \lambda \le \frac{9}{4},$$
$$'' ='' \text{ iff } \left(x = y = z = \frac{1}{3}, \lambda = \frac{9}{4}\right) \text{ or } \left(x = 0, y = z = \frac{1}{2}, \lambda = \frac{9}{4}\right)$$
$$\text{or } \left(y = 0, z = x = \frac{1}{2}, \lambda = \frac{9}{4}\right) \text{ or } \left(z = 0, x = y = \frac{1}{2}, \lambda = \frac{9}{4}\right) \text{(QED)}$$