ROMANIAN MATHEMATICAL MAGAZINE

SP.528 If $a, b, c \ge 1$ then:

$$\frac{1}{9}(a+b+c)+\frac{1}{3\sqrt{2}} \ge \frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}}+\frac{\sqrt[3]{bc-1}}{c+a+\sqrt{2}}+\frac{\sqrt[3]{ca-1}}{a+b+\sqrt{2}}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

$$RHS=\sum\frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}} \overset{AGM}{\le} \sum\frac{1}{3}\sqrt[3]{\frac{ab-1}{\sqrt{2}bc}}=\frac{1}{3}\sum\sqrt[3]{\frac{1}{\sqrt{2}}\left(a-\frac{1}{b}\right)\frac{1}{c}} \overset{AGM}{\le}$$

$$\overset{AGM}{\le} \frac{1}{3}\sum\frac{\frac{1}{\sqrt{2}}+\left(a-\frac{1}{b}\right)+\frac{1}{c}}{3}=\frac{a+b+c+\frac{3}{\sqrt{2}}}{9}=LHS$$

with equality for $b=c=\sqrt{2}, \frac{1}{\sqrt{2}}=\left(a-\frac{1}{b}\right)=\frac{1}{c}$.

Equality holds if and only if $a=b=c=\sqrt{2}$.

Solution 2 by Amir Sofi-Kosovo

$$b+c+\sqrt{2} \overset{AM-GM}{\ge} 3\sqrt[3]{\sqrt{2}bc}$$

$$\Rightarrow \frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}} \le \frac{\sqrt[3]{ab-1}}{3\sqrt[3]{\sqrt{2}bc}}=\frac{3}{9}\sqrt[3]{\left(a-\frac{1}{b}\right)\frac{1}{c}\frac{1}{\sqrt{2}}} \le \frac{a-\frac{1}{b}+\frac{1}{c}+\frac{1}{\sqrt{2}}}{9}$$

$$\frac{\sqrt[3]{bc-1}}{c+a+\sqrt{2}} \le \frac{\sqrt[3]{bc-1}}{3\sqrt[3]{\sqrt{2}ca}}=\frac{3}{9}\sqrt[3]{\left(b-\frac{1}{c}\right)\frac{1}{a}\frac{1}{\sqrt{2}}} \le \frac{b-\frac{1}{c}+\frac{1}{a}+\frac{1}{\sqrt{2}}}{9}$$

$$\frac{\sqrt[3]{ca-1}}{a+b+\sqrt{2}} \le \frac{\sqrt[3]{ca-1}}{3\sqrt[3]{\sqrt{2}ab}}=\frac{3}{9}\sqrt[3]{\left(c-\frac{1}{a}\right)\frac{1}{b}\frac{1}{\sqrt{2}}} \le \frac{c-\frac{1}{a}+\frac{1}{b}+\frac{1}{\sqrt{2}}}{9}$$

Add 3 Inequalities we get:

$$\frac{\sqrt[3]{ab-1}}{b+c+\sqrt{2}}+\frac{\sqrt[3]{bc-1}}{c+a+\sqrt{2}}+\frac{\sqrt[3]{ca-1}}{a+b+\sqrt{2}} \le \frac{1}{9}(a+b+c)+\frac{1}{3\sqrt{2}}$$