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SP.529 Let ABC be a triangle with inradius r and circumradius R and let the interior points D, E, F be chosen on the sides BC, CA, AB respectively, so that AD, BE, CF are the bisectors of the triangle ABC.Let r_A, r_B, r_C be the inradii of the triangles AEF, BFD, CDE respectively. Prove that:

$$r_A^2 + r_B^2 + r_C^2 \le \frac{3R^4}{64r^2}$$

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Solution 1 by proposer, Solution 2 by Tapas Das – India, Solution 3 by Marin Chirciu – Romania

Solution 1 by proposer

Let $\alpha = BC, b = CA, c = AB$ be the lengths of the sides and let R_A, R_B, R_C be the circumradii of the triangles AEF, BFD, CDE respectively.

Then $FE = 2R_A \cdot \sin A$. We'll prove that $FE \le \frac{2a+b+c}{8}$. By the law of cosines in triangle

$$AEF: EF^2 = AE^2 + AF^2 - 2AE \cdot AF \cdot \cos A =$$

$$= \left(\frac{bc}{a+c}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\left(\frac{bc}{a+c}\right) \cdot \left(\frac{bc}{a+b}\right) \cdot \frac{b^2+c^2-a^2}{2bc} =$$

$$= \frac{b^2c^2}{(a+c)^2} + \frac{b^2c^2}{(a+b)^2} - \frac{bc[(b-c)^2 + 2bc - a^2]}{(a+b)(a+c)} =$$

$$= b^2c^2\left(\frac{1}{a+c} - \frac{1}{a+b}\right)^2 - \frac{bc(b-c)^2}{(a+b)(a+c)} + \frac{a^2bc}{(a+b)(a+c)} =$$

$$= \frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2} - \frac{bc(b-c)^2}{(a+b)(a+c)} + \frac{a^2bc}{(a+b)(a+c)} =$$

$$= \frac{a^2bc}{(a+b)(a+c)} - \frac{bc(b-c)^2[(a+b)(a+c) - bc]}{(a+b)^2(a+c)^2} \le \frac{a^2bc}{(a+b)(a+c)} \le$$

$$\le \frac{a^2bc}{2\sqrt{ab} \cdot 2\sqrt{bc}} = \frac{a\sqrt{bc}}{4}$$

So $EF \le \frac{\sqrt{a\sqrt{bc}}}{2} = \frac{2\sqrt{a}\sqrt{\sqrt{bc}}}{4} \le \frac{a+\sqrt{bc}}{4} \le \frac{a+\frac{b+c}{2}}{4} = \frac{2a+b+c}{8}$. Namely

$EF \leq \frac{2a+b+c}{8}$ and analogs, $FD \leq \frac{a+2b+c}{8}$, $DE \leq \frac{a+b+2c}{8}$. Now, we have $2R_A \cdot \sin A \leq \frac{2a+b+c}{8}$.

But $R_A \geq 2r_A$ (EULER), so $4r_A \cdot \frac{a}{2R} \leq \frac{2a+b+c}{8} \Leftrightarrow r_A \leq \frac{R}{16}\left(2+\frac{b}{a}+\frac{c}{a}\right)$. Similarly,

$r_B \leq \frac{R}{16}\left(2+\frac{c}{b}+\frac{a}{b}\right)$, and $r_C \leq \frac{R}{16}\left(2+\frac{a}{c}+\frac{b}{c}\right)$. Now, we have

$$r_A^2+r_B^2+r_C^2 \leq \frac{R^2}{256}\left[\left(2+\frac{b}{a}+\frac{c}{a}\right)^2+\left(2+\frac{c}{b}+\frac{a}{b}\right)^2+\left(2+\frac{a}{c}+\frac{b}{c}\right)^2\right]=$$

$$=\frac{R^2}{256}\left[\begin{array}{c}4+4\left(\frac{b}{a}+\frac{c}{a}\right)+\frac{b^2}{a^2}+\frac{c^2}{a^2}+\frac{2bc}{a^2}+4+4\left(\frac{c}{b}+\frac{a}{b}\right)+\frac{c^2}{b^2}+\frac{a^2}{b^2}+\frac{2ca}{b^2}+\\+4+4\left(\frac{a}{c}+\frac{b}{c}\right)+\frac{a^2}{c^2}+\frac{b^2}{c^2}+\frac{2ab}{c^2}\end{array}\right]=$$

$$=\frac{R^2}{256}\left[\begin{array}{c}12+4\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right)+\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\\+\left(\frac{2bc}{a^2}+\frac{2ca}{b^2}+\frac{2ab}{c^2}\right)\end{array}\right]\leq$$

$$\frac{R^2}{256}\left[\begin{array}{c}12+4\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right)+\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)+\\+\frac{b^2+c^2}{a^2}+\frac{c^2+a^2}{b^2}+\frac{a^2+b^2}{c^2}\end{array}\right]=$$

$$\frac{R^2}{256}\left[\begin{array}{c}12+4\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right)+2\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+2\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)+\\+2\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)\end{array}\right]$$

It is well-known that: $\frac{a}{b}+\frac{b}{a} \leq \frac{R}{r}, \frac{b}{c}+\frac{c}{b} \leq \frac{R}{r}$, and $\frac{c}{a}+\frac{a}{c} \leq \frac{R}{r}$. So

$\frac{a^2}{b^2}+\frac{b^2}{a^2} \leq \frac{R^2}{r^2}-2, \frac{b^2}{c^2}+\frac{c^2}{b^2} \leq \frac{R^2}{r^2}-2$, and $\frac{c^2}{a^2}+\frac{a^2}{c^2} \leq \frac{R^2}{r^2}-2$

Namely:

$$r_A^2+r_B^2+r_C^2 \leq \frac{R^2}{256}\cdot\left[12+4\cdot 3\frac{R}{r}+6\left(\frac{R^2}{r^2}-2\right)\right]=$$

$$\frac{R^2}{256}\cdot 6\left(\frac{2R}{r}+\frac{R^2}{r^2}\right)=\frac{3R^2}{128}\cdot\frac{R}{r}\left(2+\frac{R}{r}\right)=\frac{3R^2}{128}\cdot\frac{R}{r}\cdot\frac{2r+R}{r}\overset{EULER}{\leq}\frac{3R^3}{128r^2}(R+R)=\frac{3R^4}{64r^2}$$

So $r_A^2+r_B^2+r_C^2 \leq \frac{3R^4}{64r^2}$. Equality holds iff the triangle ABC is equilateral.

Solution 2 by Tapas Das – India

R_A, R_B, R_c are the circumradius of $\Delta AEF, \Delta BFD, \Delta CDE$ respectively

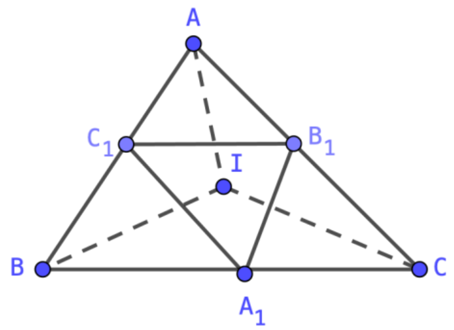
$$\therefore R_A \le \frac{B_1C_1}{2\sin A} \quad \therefore R_A^2 \le \frac{B_1C_1^2}{4\sin^2 A} \quad \therefore R_A^2 \le \frac{(a^2bc)^{\frac{1}{2}}}{16\sin^2 A} = \frac{R^2\sqrt{bc}}{4a}$$

$$\therefore r_A^2 + r_B^2 + r_C^2 \le \left(\frac{R_A}{2}\right)^2 + \left(\frac{R_B}{2}\right)^2 + \left(\frac{R_C}{2}\right)^2 = \frac{1}{4}\left[R_A^2 + R_B^2 + R_C^2\right]$$

$$\le \frac{1}{4}R^2\left[\frac{\sqrt{bc}}{4a} + \frac{\sqrt{ca}}{4b} + \frac{\sqrt{ab}}{4c}\right] = \frac{1}{16}R^2\left[\frac{\sqrt{bc}}{a} + \frac{\sqrt{ca}}{b} + \frac{\sqrt{ab}}{c}\right] \overset{CBS}{\le} \frac{1}{16}R^2\sqrt{\left(\sum bc\right)\cdot\sum\frac{1}{a^2}}$$

$$\le \frac{R^2}{16}\sqrt{\sum a^2 \cdot \sum\frac{1}{a^2}} \quad \left[\because \sum bc \le \sum a^2\right]$$

$$\overset{Steining\ Leibniz}{\le} \frac{R^2}{16}\sqrt{\frac{9R^2}{4r^2}} = \frac{R^2}{16}\cdot\frac{3R}{2r} = \frac{3R^3}{32r} = \frac{3R^4}{32rR} \overset{Euler}{\le} \frac{3R^4}{64r^2}$$



$$AB_1 = \frac{bc}{c+a}, AC_1 = \frac{bc}{a+b}$$

$$B_1C_1^2 = AC_1^2 + AB_1^2 - 2AC_1AB_1\cos A$$

$$= \left(\frac{bc}{c+a}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right)\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{bc[bc(a+b)^2 + bc(c+a)^2 - (a+b)(a+c)(b^2+c^2-a^2)]}{(a+b)^2(a+c)^2}$$

$$= \frac{bc[a^2(a+b)(a+c) - (b^2+c^2)(a+b)(a+c) + bc(a+b)^2 + bc(c+a)^2]}{(a+b)^2(a+c)^2}$$

$$= \frac{bc[a^2(a+b)(a+c) - a(b-c)^2(a+b+c)]}{(a+b)^2(a+c)^2}$$

$$\leq \frac{bca^2(a+b)(a+c)}{(a+b)^2(a+c)^2} = \frac{a^2bc}{(a+b)(a+c)}$$

Note: $bc(a+b)^2 + bc(c+a)^2 - (b^2+c^2)(a+b)(a+c)$

$$= bc[(b-c)^2 + 2(a+b)(a+c)] - [(b-c)^2 + 2bc](a+b)(a+c)$$

$$= bc(b-c)^2 + 2bc(a+b)(a+c) - (b-c)^2(a+b)(a+c) - 2bc(a+b)(a+c)$$

$$= (b-c)^2[bc - (a+b)(a+c)]$$

$$= (b-c)^2[bc - a^2 - ac - ba - bc] = -a(b-c)^2(a+b+c)$$

$$\therefore B_1C_1^2 \leq \frac{a^2bc}{(a+b)(a+c)} \leq \frac{a^2bc}{4a\sqrt{bc}} \quad \text{(AM-GM)} = \frac{a\sqrt{bc}}{4} \therefore B_1C_1 \leq \frac{(a^2bc)^{\frac{1}{4}}}{2}$$

$$\therefore R_A = \frac{B_1C_1}{2\sin A}$$

Solution 3 by Marin Chirciu – Romania

Lemma.

In $\Delta ABC, AD, BE, CF$ – internal bisectors, r_A – inradii ΔAEF:

$$r_A^2 \leq \frac{R^2bc}{4(a+b)(a+c)}$$

Proof.

Let R_A, R_B, R_C – circumradii $\Delta AEF, \Delta BFD, \Delta CDE$.

$$R_A = \frac{EF}{2\sin A}$$

Lemma 1.

In $\Delta ABC, AD, BE, CF$ – internal bisectors

$$EF^2 \leq \frac{a^2bc}{(a+b)(a+c)}$$

Proof.

With bisector theorem we have $AD = \frac{bc}{a+c}, AF = \frac{bc}{a+b}$.

Using cosine theorem in ΔAEF we obtain:

$$EF^2 = AE^2 + AF^2 - 2AE \cdot AF \cdot \cos A =$$

$$= \left(\frac{bc}{a+c}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\frac{bc}{a+c} \cdot \frac{bc}{a+b} \cdot \frac{b^2+c^2-a^2}{2bc} =$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc(b^2+c^2-a^2)}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{bc[2bc+(b-c)^2-a^2]}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{2b^2c^2}{(a+b)(a+c)}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\underbrace{\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{2b^2c^2}{(a+b)(a+c)}}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=b^2c^2\left(\frac{1}{a+c}-\frac{1}{a+b}\right)^2-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{b^2c^2(b-c)^2}{(a+b)^2(a+c)^2}-\frac{bc(b-c)^2}{(a+b)(a+c)}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{bc(b-c)^2[bc-(a+b)(a+c)]}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{bc(b-c)^2(-a^2-ab-ac)}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}=$$

$$=\frac{-abc(b-c)^2(a+b+c)}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}\le\frac{a^2bc}{(a+b)(a+c)}$$

Equality holds if and only if the triangle is equilateral.

We have $r_A\le\frac{R_A}{2}$ **and** $R_A=\frac{EF}{2\sin A}\Rightarrow r_A=\frac{EF}{4\sin A}\Rightarrow$

$$r_a^2=\frac{EF^2}{16\sin^2 A}=\frac{EF^2}{16\,\frac{a^2}{4R^2}}=\frac{R^2EF^2}{4a^2}\overset{Lemma\ 1}{\le}\frac{R^2\cdot\frac{a^2bc}{(a+b)(a+c)}}{4a^2}=\frac{R^2bc}{4(a+b)(a+c)}$$

Let's get back to the main problem.

Using Lemma $r_A^2\le\frac{R^2bc}{4(a+b)(a+c)}$ **we obtain:**

$$\sum r_A^2\overset{Lemma}{\le}\sum\frac{R^2bc}{4(a+b)(a+c)}=R^2\frac{\sum bc(b+c)}{4\prod(b+c)}=$$

$$=\frac{R^2}{4}\cdot\frac{2p(p^2-r^2-2Rr)}{2p(p^2+r^2+2Rr)}=\frac{R^2(p^2+r^2-2Rr)}{4(p^2+r^2+2Rr)}$$

We prove that:

$$\frac{R^2(p^2+r^2-2Rr)}{4(p^2+r^2+2Rr)} \le \frac{3R^4}{64r^2} \Leftrightarrow 16r^2(p^2+r^2-2Rr) \le 3R^2(p^2+r^2+2Rr) \Leftrightarrow$$

$$\Leftrightarrow p^2(3R^2-16r^2)+r(6R^3+3R^2r+32Rr^2-16r^3) \ge 0$$

We distinguish the cases:

Case 1. If $(3R^2-16r^2) \ge 0$ the inequality is obvious.

Case 2. If $(3R^2-16r^2) < 0$ the inequality can be written:

$r(6R^3+3R^2r+32Rr^2-16r^3) \ge p^2(16r^2-3R^2)$, which follows from Gerretsen's inequality: $p^2 \le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(6R^3+3R^2r+32Rr^2-16r^3) \ge (4R^2+4Rr+3r^2)(16r^2-3R^2) \Leftrightarrow$$

$$\Leftrightarrow 6R^4+9R^3r-26R^2r^2-16Rr^3-32r^4 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(6R^3+21Rr^2+16Rr^2+16r^3) \ge 0$, see $R \ge 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.