SP.530. Let a, b, c be the lengths of the sides of a triangle with inradius circumradius R. Prove that:

$$\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)R}} \le \frac{1}{\sqrt[n]{m\cdot a+b}} + \frac{1}{\sqrt[n]{m\cdot b+c}} + \frac{1}{\sqrt[n]{m\cdot c+a}} \le \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot 2r}}$$

for all integers $m \ge 0$ and $n \ge 1$.

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by proposer

For the right inequality, first will prove that

$\frac{1}{m\cdot a+b} \le \frac{1}{(m+1)^2}\left(\frac{m}{a}+\frac{1}{b}\right)$. We have

$$\frac{1}{(m+1)^2}\left(\frac{m}{a}+\frac{1}{b}\right) - \frac{1}{m\cdot a+b} = \frac{(a+mb)(ma+b)-(m+1)^2ab}{(m+1)^2ab(ma+b)} =$$

$\frac{m(a-b)^2}{(m+1)^2ab(ma+b)} \ge 0$. So

$\frac{1}{\sqrt[n]{ma+b}} \le \frac{1}{\sqrt[n]{(m+1)^2}}\sqrt[n]{\frac{m}{a}+\frac{1}{b}}$. Similarly

$$\frac{1}{\sqrt[n]{mb+c}} \le \frac{1}{\sqrt[n]{(m+1)^2}}\sqrt[n]{\frac{m}{b}+\frac{1}{c}}, \frac{1}{\sqrt[n]{mc+a}} \le \frac{1}{\sqrt[n]{(m+1)^2}}\sqrt[n]{\frac{m}{c}+\frac{1}{a}}$$

Adding up these inequalities, we have

$$\frac{1}{\sqrt[n]{ma+b}} + \frac{1}{\sqrt[n]{mb+c}} + \frac{1}{\sqrt[n]{mc+a}} \le \frac{1}{\sqrt[n]{(m+1)^2}}\left(\sqrt[n]{\frac{m}{a}+\frac{1}{b}} + \sqrt[n]{\frac{m}{b}+\frac{1}{c}} + \sqrt[n]{\frac{m}{c}+\frac{1}{a}}\right)$$

We know that $\left(\frac{x+y+z}{3}\right)^n \le \frac{x^n+y^n+z^n}{3}$, $(x, y, z > 0)$. So for

$x = \sqrt[n]{\frac{m}{a}+\frac{1}{b}}, y = \sqrt[n]{\frac{m}{b}+\frac{1}{c}}, z = \sqrt[n]{\frac{m}{c}+\frac{1}{a}}$, we get

$$\left(\frac{\sqrt[n]{\frac{m}{a}+\frac{1}{b}}+\sqrt[n]{\frac{m}{b}+\frac{1}{c}}+\sqrt[n]{\frac{m}{c}+\frac{1}{a}}}{3}\right)^n \le \frac{(m+1)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{3} \text{ so}$$

$$\sqrt[n]{\frac{m}{a}+\frac{1}{b}} + \sqrt[n]{\frac{m}{b}+\frac{1}{c}} + \sqrt[n]{\frac{m}{c}+\frac{1}{a}} \le \frac{3}{\sqrt[n]{3}}\sqrt[n]{m+1}\cdot\sqrt[n]{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

We know that $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\le 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$ and

$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}$ (easy proof), so $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2r}$. Now we have

$$\sqrt[n]{\frac{m}{a}+\frac{1}{b}}+\sqrt[n]{\frac{m}{b}+\frac{1}{c}}+\sqrt[n]{\frac{m}{c}+\frac{1}{a}}\le\frac{1}{\sqrt[n]{(m+1)^2}}\cdot\frac{3}{\sqrt[n]{3}}\sqrt[n]{m+1}\cdot\frac{\sqrt[n]{\sqrt{3}}}{\sqrt[n]{2r}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot 2r}}$$

For the left inequality, we use the AM-GM inequality, we have

$$\frac{1}{\sqrt[n]{ma+b}}+\frac{1}{\sqrt[n]{mb+c}}+\frac{1}{\sqrt[n]{mc+a}}\ge\frac{3}{\sqrt[3]{\sqrt[n]{(ma+b)(mb+c)(mc+a)}}}=$$

$$\frac{3}{\sqrt[n]{\sqrt[3]{(ma+b)(mb+c)(mc+a)}}}\ge\frac{3}{\sqrt[n]{\frac{(ma+b)+(mb+c)+(mc+a)}{3}}}=$$

$$\frac{3\sqrt[n]{3}}{\sqrt[n]{(m+1)(a+b+c)}}\ge\frac{3\sqrt[n]{3}}{\sqrt[n]{m+1}\cdot\sqrt[n]{3\sqrt{3}R}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)R}}$$

$$\left(a+b+c\le 3\sqrt{3}R\right)$$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Tapas Das – India

$$\frac{1}{\sqrt[n]{ma+b}}+\frac{1}{\sqrt[n]{mb+c}}+\frac{1}{\sqrt[n]{mc+a}}=\frac{(1)^{\frac{1}{n}+1}}{\sqrt[n]{ma+b}}+\frac{(1)^{\frac{1}{n}+1}}{\sqrt[n]{mb+c}}+\frac{(1)^{\frac{1}{n}+1}}{\sqrt[n]{mc+a}}\ge$$

$$\overset{Radon}{\ge}\frac{(1+1+1)^{\frac{1}{n}+1}}{\left[m(\sum a)+\sum a\right]^{\frac{1}{n}}}=\frac{3^{\frac{1}{n}+1}}{(\sum a)^{\frac{1}{n}}\cdot(m+1)^{\frac{1}{n}}}=\frac{3^{\frac{1}{n}+1}}{(2s)^{\frac{1}{n}}\cdot(m+1)^{\frac{1}{n}}}\ge$$

$$\overset{Mitrinovic}{\ge}\frac{3^{\frac{1}{n}+1}}{(3\sqrt{3}R)^{\frac{1}{n}}(m+1)^{\frac{1}{n}}}=\frac{3^{\frac{1}{n}+1-\frac{3}{2n}}}{\sqrt{(m+1)\cdot R}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt{(m+1)R}}$$

$$\frac{1}{am+b}+\frac{1}{bm+c}+\frac{1}{cm+a}$$

$$=\frac{(bm+c)(cm+a)+(am+b)(cm+a)+(am+b)(bm+c)}{(am+b)(bm+c)(cm+a)}$$

$$\overset{Holder}{\le}\frac{\frac{[(bm+c)(cm+a)+(am+b)]^2}{3}}{\left[(abcm^3)^{\frac{1}{3}}+(abc)^{\frac{1}{3}}\right]^3}=\frac{(a+b+c)^2(m+1)^2}{(abc)(m+1)^3\cdot 3}$$

$$=\frac{4s^2}{3abc(m+1)}=\frac{4s^2}{3(4Rrs)(m+1)}=\frac{s}{3Rr(cm+1)}$$

$$\overset{Mitrinovic}{\leq}\frac{3\sqrt{3}R}{2\cdot 3Rr(m+1)}=\frac{3^{\frac{1}{2}}}{(m+1)2r}\quad (1)$$

$$\therefore \frac{1}{\sqrt[n]{am+b}}+\frac{1}{\sqrt[n]{bm+c}}+\frac{1}{\sqrt[n]{cm+a}}\overset{Holder}{\leq}\sqrt[n]{(1+1+1)^{n-1}\sum\frac{1}{am+b}}$$

$$\leq\sqrt[n]{3^{n-1}\cdot\frac{3^{\frac{1}{2}}}{(m+1)2r}}\text{, using (1)}=\sqrt[n]{\frac{3^{n-\frac{1}{2}}}{(m+1)2r}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)2r}}$$

Solution 3 by Marin Chirciu – Romania

RHS

$$\sum\frac{1}{\sqrt[n]{m\cdot a+b}}\overset{Holder}{\leq}\sqrt[n]{3^{n-1}\sum\frac{1}{m\cdot a+b}}\overset{(1)}{\leq}\sqrt[n]{3^{n-1}\cdot\frac{1}{m+1}\cdot\frac{\sqrt{3}}{2r}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot 2r}}$$

where (1) $\Leftrightarrow \sum\frac{1}{m\cdot a+b}\leq\frac{1}{m+1}\cdot\frac{\sqrt{3}}{2r}$, see:

$$\sum\frac{1}{m\cdot a+b}\overset{CS}{\leq}\frac{1}{(m+1)^2}\sum\left(\frac{m}{a}+\frac{1}{b}\right)=\frac{1}{(m+1)^2}(m+1)\sum\frac{1}{a}=$$

$$=\frac{1}{m+1}\sum\frac{1}{a}\overset{Leunberger}{\leq}\frac{1}{m+1}\cdot\frac{\sqrt{3}}{2r}$$

Equality holds if and only if the triangle is equilateral.

LHS

$$\sum\frac{1}{\sqrt[n]{m\cdot a+b}}\overset{Holder}{\geq}\frac{9}{\sum\sqrt[n]{m\cdot a+b}}\overset{(2)}{\geq}\frac{9}{\sqrt[n]{3^{n+\frac{1}{2}}(m+1)\cdot R}}=\frac{3^2}{3^{1+\frac{1}{2n}}\sqrt[n]{(m+1)\cdot R}}=\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot R}},$$

Where (2) $\Leftrightarrow \sum\sqrt[n]{m\cdot a+b}\leq\sqrt[n]{3^{n+\frac{1}{2}}(m+1)\cdot R}$, see:

$$\sum\sqrt[n]{m\cdot a+b}\overset{Holder}{\leq}\sqrt[n]{3^{n-1}\sum(m\cdot a+b)}=\sqrt[n]{3^{n-1}(m+1)\sum a}=$$

$$=\sqrt[n]{3^{n-1}(m+1)\cdot 2p}\overset{Mitrinovic}{\leq}$$

$$\overset{Mitrinovic}{\leq}\sqrt[n]{3^{n-1}(m+1)\cdot 3\sqrt{3}R}=\sqrt[n]{3^{n+\frac{1}{2}}(m+1)\cdot R}$$

Equality holds if and only if the triangle is equilateral.