ROMANIAN MATHEMATICAL MAGAZINE

SP.531 If $a,b,c \geq 1$, then :

$$\sqrt{\frac{ab+bc+ca}{3}}-\sqrt[3]{abc} \geq \sqrt{\frac{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}{3}}-\frac{1}{\sqrt[3]{abc}}.$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by proposer

We write the inequality in the form $E(a,b,c) \geq 0$.
Without loss of generality, we may assume that $a \geq b \geq c \geq 1$.
We shall prove that: $E(a,b,c) \geq E(a,\sqrt{bc},\sqrt{bc}) \geq 0$.

a) We will prove the inequality $E(a,b,c) \geq E(a,\sqrt{bc},\sqrt{bc})$.
The inequality can be written as follows:

$$\sqrt{\frac{ab+bc+ca}{3}}-\sqrt{\frac{bc+2a\sqrt{bc}}{3}} \geq \sqrt{\frac{a+b+c}{3abc}}-\sqrt{\frac{a+2\sqrt{bc}}{3abc}}$$

Or, equivalently:

$$\frac{\frac{1}{3}a\left(\sqrt{b}-\sqrt{c}\right)^2}{\sqrt{\frac{ab+bc+ca}{3}}+\sqrt{\frac{bc+2a\sqrt{bc}}{3}}} \geq \frac{\frac{1}{3}\frac{1}{abc}\left(\sqrt{b}-\sqrt{c}\right)^2}{\sqrt{\frac{a+b+c}{3abc}}+\sqrt{\frac{a+2\sqrt{bc}}{3abc}}}$$

To prove the inequality $E(a,b,c) \geq E(a,\sqrt{bc},\sqrt{bc})$ **it is enough to prove that:**

$$\left(\sqrt{b}-\sqrt{c}\right)^2\frac{1}{\sqrt{\frac{ab+bc+ca}{a^2}}+\sqrt{\frac{bc+2a\sqrt{bc}}{a^2}}} \geq \left(\sqrt{b}-\sqrt{c}\right)^2\frac{1}{\sqrt{abc(a+b+c)}+\sqrt{abc\left(a+2\sqrt{bc}\right)}}$$

For $b=c$ **this relationship it is true (case of equality).**
For $b \neq c$ **we will show that:**

$\frac{ab+bc+ca}{a^2} \leq abc(a+b+c)$ **(inequality 1) and** $\frac{bc+2a\sqrt{bc}}{a^2} \leq abc\left(a+2\sqrt{bc}\right)$ **(inequality 2)**

We prove inequalities (1) and (2). We have:

$ab+bc+ca \leq ab+a^2+ca = a(a+b+c) \leq a^3bc(a+b+c)$ **and:**

$$bc+2a\sqrt{bc} = a\left(\frac{bc}{a}+2\sqrt{bc}\right) \leq a\left(a+2\sqrt{bc}\right) \leq a^3bc\left(a+2\sqrt{bc}\right)$$

Thus, the inequality $E(a,b,c) \geq E(a,\sqrt{bc},\sqrt{bc})$ **is proved.**

b) We will prove the inequality: $E(a,\sqrt{bc},\sqrt{bc}) \geq 0$ **. Let us denote:** $x=\sqrt{bc}$, $a \geq x \geq 1$.
We have to show that:

$\sqrt{\frac{x^2+2ax}{3}}-\sqrt[3]{ax^2} \geq \sqrt{\frac{a+2x}{3ax^2}}-\frac{1}{\sqrt[3]{ax^2}}$ **. But, we have:** $x^2 \geq 1$.

Since both sides of the above inequality are nonnegative, it is enough to prove the homogeneous inequality:

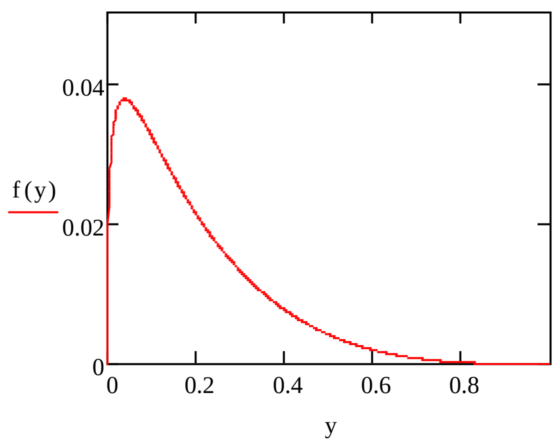
$\sqrt{\frac{x^2+2ax}{3}}-\sqrt[3]{ax^2}\geq x^2\left(\sqrt{\frac{a+2x}{3ax^2}}-\frac{1}{\sqrt[3]{ax^2}}\right)$**.This inequality is equivalently written:**

$\sqrt{\frac{1}{3}\left(y^2+2y\right)}-\sqrt[3]{y^2}\geq\sqrt{\frac{1}{3}\left(y^2+2y^3\right)}-\sqrt[3]{y^4}$ **, where:** $y=\frac{x}{a}$**,** $0<y\leq 1$

To prove this inequality,we study the variation and we draw the graph of the function:

$f(y)=\sqrt{\frac{1}{3}\left(y^2+2y\right)}-\sqrt[3]{y^2}-\sqrt{\frac{1}{3}\left(y^2+2y^3\right)}+\sqrt[3]{y^4}$ **,** $y\in(0,1]$

We obtain:



It follows that we have $f(y)\geq 0$**,** $y\in(0,1]$ **(we have** $f(1)=0$**).Thus, the inequality** $E(a,\sqrt{bc},\sqrt{bc})\geq 0$ **is proved.So, the inequality** $E(a,b,c)\geq 0$ **required in the statement of the problem is proved.**

Remark: The expression:

$$M=\sqrt{\frac{ab+bc+ca}{3}}$$

is an elementary symmetric mean of the numbers a, b, c. We have: $M\geq G=\sqrt[3]{abc}$ **.**

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $a\geq b\geq c$. Let $x := \sqrt{bc}$ and we will write the given inequality as

$$E(a,b,c)=\sqrt{\frac{ab+bc+ca}{3}}-\sqrt[3]{abc}-\sqrt{\frac{a+b+c}{3abc}}+\frac{1}{\sqrt[3]{abc}}\geq 0.$$

We will first prove that $E(a,b,c) \geq E(a,x,x)$. The inequality is successively equivalent to

$$\sqrt{\frac{ab+bc+ca}{3}}-\sqrt{\frac{a+b+c}{3abc}} \geq \sqrt{\frac{2a\sqrt{bc}+bc}{3}}-\sqrt{\frac{a+2\sqrt{bc}}{3abc}}$$

$$\left(\sqrt{ab+bc+ca}-\sqrt{2a\sqrt{bc}+bc}\right)\sqrt{abc} \geq \sqrt{a+b+c}-\sqrt{a+2\sqrt{bc}}$$

$$\frac{a\left(\sqrt{b}-\sqrt{c}\right)^2.\sqrt{abc}}{\sqrt{ab+bc+ca}+\sqrt{2a\sqrt{bc}+bc}} \geq \frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{\sqrt{a+b+c}+\sqrt{a+2\sqrt{bc}}}$$

$$\sqrt{a^3bc(a+b+c)}+\sqrt{a^3bc\left(a+2\sqrt{bc}\right)} \geq \sqrt{ab+bc+ca}+\sqrt{2a\sqrt{bc}+bc},$$

which is true because $a^3bc(a+b+c) \geq ab+bc+ca$ and $a^3bc\left(a+2\sqrt{bc}\right) \geq 2a\sqrt{bc}+bc$.

So it suffices to prove that $E(a,x,x) \geq 0$, and since $x \geq 1$, it suffices to prove that

$$\sqrt{\frac{2ax+x^2}{3}}-\sqrt[3]{ax^2} \geq x^2\left(\sqrt{\frac{a+2x}{3ax^2}}-\frac{1}{\sqrt[3]{ax^2}}\right).$$

Setting $t=\frac{a}{x} \geq 1$. The inequality is equivalent to

$$\sqrt{\frac{2t+1}{3}}-\sqrt[3]{t} \geq \sqrt{\frac{t+2}{3t}}-\frac{1}{\sqrt[3]{t}} \Leftrightarrow \sqrt{t(2t+1)}-\sqrt{t+2} \geq \sqrt{3}.\sqrt[6]{t}\left(\sqrt[3]{t^2}-1\right)$$

$$\Leftrightarrow \frac{2(t^2-1)}{\sqrt{t(2t+1)}+\sqrt{t+2}} \geq \sqrt{3}.\sqrt[6]{t}\left(\sqrt[3]{t^2}-1\right)$$

$$\overset{t\geq 1}{\Leftrightarrow} 2\left(\sqrt[3]{t^4}+\sqrt[3]{t^2}+1\right) \geq \sqrt[6]{t}\left(\sqrt{3t(2t+1)}+\sqrt{3(t+2)}\right).$$

Let $y=\sqrt[6]{t}$. By $\mathrm{AM-GM}$ inequality, we have

$$RHS=\sqrt[6]{t}\left(\sqrt{3t(2t+1)}+\sqrt{3(t+2)}\right) \leq \sqrt[6]{t}\left(\frac{3t+(2t+1)}{2}+\frac{3+(t+2)}{2}\right)$$

$$=3y(y^6+1) \overset{?}{\leq} LHS$$

$$2(y^8+y^4+1) \geq 3y(y^6+1) \Leftrightarrow (y-1)^2\left(2y^6+y^5-y^3+y+1\right) \geq 0,$$

which is true and the proof is complete. Equality holds iff $a=b=c=1$.