ROMANIAN MATHEMATICAL MAGAZINE

SP.532 Prove that in any right triangle with the cathetus b and c we have the inequality:

$$r \le \frac{2-\sqrt{2}}{4}(b+c), \text{ where } r \text{ is the inradii of the triangle.}$$

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Solution 1 by proposers, Solution 2 by Marin Chirciu – Romania

Solution 1 by proposers

We have $r = \frac{S}{p}, S = \frac{bc}{2} \Rightarrow bc = r(a+b+c) \Rightarrow a = \frac{bc-rb-rc}{r} = \frac{bc}{r} - b - c.$

But $b^2 + c^2 = a^2 \Rightarrow b^2 + c^2 = \left(\frac{bc}{r} - b - c\right)^2$

$$r^2b^2 + r^2c^2 = b^2c^2 + b^2r^2 + c^2r^2 - 2b^2cr - 2bc^2r + 2bcr^2$$

$$b^2c^2 - 2b^2cr - 2bc^2r + 2bcr^2 = 0|: bc$$

$$bc - 2br - 2cr + 2r^2 = 0, \qquad b(c-2r) - 2r(c-2r) = 2r^2$$

$$(b-2r)(c-2r) = 2r^2 \quad (*)$$

But $a + b > c \Leftrightarrow a + b + c > 2c \Leftrightarrow 1 > \frac{2c}{a+b+c}$

$$\Leftrightarrow b > \frac{2bc}{a+b+c} = \frac{2r(a+b+c)}{a+b+c} = 2r \Rightarrow b > 2r$$

$\Rightarrow b - 2r > 0$. Analogous we obtain $c - 2r > 0$

As $Mg \le Ma \Rightarrow$ from inequality (*) the result

$$r\sqrt{2} = \sqrt{(b-2r)(c-2r)} \le \frac{b-2r+c-2r}{2} \Leftrightarrow 2r\sqrt{2} + 4r \le b + c \Leftrightarrow$$

$$2r(\sqrt{2}+2) \le b + c \Leftrightarrow r \le \frac{1}{2(2+\sqrt{2})}(b+c) \Leftrightarrow r \le \frac{2-\sqrt{2}}{4}(b+c)$$

We have equality $\Leftrightarrow$ the isosceles right triangle.

Solution 2 by Marin Chirciu – Romania

Using $r = \frac{b+c-a}{2}$ the inequality from enunciation can be written $\frac{b+c-a}{2} \le \frac{2-\sqrt{2}}{4}(b+c) \Leftrightarrow$

$$\Leftrightarrow 2a \ge \sqrt{2}(b+c) \Leftrightarrow 2a^2 \ge (b+c)^2 \Leftrightarrow 2(b^2+c^2) \ge (b+c)^2 \Leftrightarrow (b-c)^2 \ge 0$$

with equality for $b = c$.

Equality holds if and only if the triangle is right-angles isosceles.