ROMANIAN MATHEMATICAL MAGAZINE

**SP. 533 Prove that $k=\frac{4}{5}$ is the largest positive value of the constant $k$ such that**

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}-5\geq k(a+b+c+d+e-5)$$

**for any positive real numbers $a,b,c,d,e$ satisfying**

$$ab+bc+cd+de+ea=5$$

*Proposed by Vasile Cîrtoaje – Romania*

*Solution by proposer*

Setting: $a=x^2, b=e=\frac{m}{x^2}, c=d=\frac{1}{x}, m,x>0$,

from the equality constraint $ab+bc+cd+de+ea=5$, we get

$$2m+\frac{2m}{x^3}+\frac{1}{x^2}=5,\quad m=\frac{x(5x^2-1)}{2(x^3+1)},$$

and the desired inequality becomes

$$\frac{1}{x^2}+\frac{2x^2}{m}+2x-5\geq k\left(x^2+\frac{2m}{x^2}+\frac{2}{x}-5\right),$$

$$\frac{1}{x^2}+\frac{4x(x^3+1)}{5x^2-1}+2x-5\geq k\left(x^2+\frac{5x^2-1}{x^4+x}+\frac{2}{x}-5\right).$$

For $x\to\infty$, this inequality leads to the necessary condition $\frac{4}{5}\geq k$.

Further, we need to prove the inequality for $k=\frac{4}{5}$, i.e. to show that $E(a,b,c,d,e)\geq 0$,

where $E=5\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)-4(a+b+c+d+e)-5$.

Let $T=(T_1,T_2,T_3,T_4,T_5)$ and $t=(t_1,t_2,t_3,t_4,t_5)$ be two decreasing sequences of nonnegative numbers. By Karamata majorization inequality applied to the convex function $f(x)=e^x$, if

$T_1\geq t_1, T_1T_2\geq t_1t_2, T_1T_2T_3\geq t_1t_2t_3, T_1T_2T_3T_4\geq t_1t_2t_3t_4$ and $T_1T_2T_3T_4T_5=t_1t_2t_3t_4t_5$, then

$$T_1+T_2+T_3+T_4+T_5\geq t_1+t_2+t_3+t_4+t_5.$$

**Let $(x_1,x_2,x_3,x_4,x_5)$ be a permutation of $(a,b,c,d,e)$ such that $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5$. According to Karamata's inequality, the largest cyclic sum of five terms $x_ix_j$, where $i \neq j$ and each $x_i$ appears twice, is $S_1 = x_1x_2 + x_1x_3 + x_2x_4 + x_3x_5 + x_4x_5$.**

**As a consequence, the smallest sum is**

$$S_2 = \sum_{1\leq i<j\leq 5} x_ix_j - S_1 = x_1x_5 + x_1x_4 + x_2x_5 + x_2x_3 + x_3x_4.$$

**Since $E(a,b,c,d,e) = E(x_1,x_2,x_3,x_4,x_5)$ and $ab+bc+cd+de+ea=5$ involves $S_2 \leq 5$, to prove that $E(a,b,c,d,e) \geq 0$ it suffices to show that $S_2 \leq 5$ involves $E(x_1,x_2,x_3,x_4,x_5) \geq 0$. Since increasing all the $x_i$ by the same multiplicative factor increases the sum $S_2$ and decreases the function $E(x_1,x_2,x_3,x_4,x_5)$, we may consider $S_2 = 5$. So, we need to show that $E(a,b,c,d,e) \geq 0$ for**

$$ae+ad+be+bc+cd=5,\quad a\geq b\geq c\geq d\geq e>0.$$

**Denote: $x=\frac{a+b}{2}, y=\frac{d+e}{2}, a\geq x\geq b\geq c\geq d\geq y\geq e.$**

**Replacing $a$ and $e$ with $2x-b$ and $2y-d$, respectively, we have**

$$5 = a(d+e)+be+bc+cd = 2(2x-b)y+b(2y-d)+bc+cd =$$
$$= 4xy+bc-(b-c)d.$$

**From this, we get: $5 \geq 4xy+bc-(b-c)c = 4xy+c^2$,**

**Hence $4xy \leq 5-c^2,\ \ c\leq\sqrt{5}$ and**

$$5 = 4xy+bc-(b-c)d \leq 4xy+bc-(b-c)y = 4xy+b(c-y)+cy$$
$$\leq 4xy+x(c-y)+cy = 3xy+c(x+y) \leq \frac{3}{4}(5-c^2)+c(x+y),$$

**hence: $4c(x+y)\geq 3c^2+5$**

**By the AM-HM inequality, we have**

$$\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}=\frac{2}{x},\quad \frac{1}{d}+\frac{1}{e}\geq\frac{2}{y}.$$

**Thus, it suffices to show that the conditions**

$$4xy\leq 5-c^2,\quad 4c(x+y)\geq 3c^2+5,\quad x\geq c\geq y>0$$

**involve**

$$5\left(\frac{2}{x}+\frac{2}{y}+\frac{1}{c}\right) \ge 4(2x+2y+c)+5,$$

**that is**

$$2(x+y)\left(\frac{5}{xy}-4\right)+\frac{5}{c}-4c-5 \ge 0.$$

**Since**

$$\frac{5}{xy}-4 \ge \frac{20}{5-c^2}-4 = \frac{4c^2}{5-c^2},$$

**it suffices to show that:** $\frac{8(x+y)c^2}{5-c^2}+\frac{5}{c}-4c-5 \ge 0.$

**Indeed,**

$$\frac{8(x+y)c^2}{5-c^2}+\frac{5}{c}-4c-5 \ge \frac{2c(3c^2+5)}{5-c^2}+\frac{5}{c}-4c-5$$

$$=\frac{5(2c^4+c^3-3c^2-5c+5)}{c(5-c^2)} = \frac{5(c-1)^2(2c^2+5c+5)}{c(5-c^2)} \ge 0$$

**The proof is completed. For** $k=\frac{4}{5}$**, the equality occurs when** $a=b=c=d=e=1.$