SP.534 If the lengths a,b,c of the sides of a triangle are the roots of the equation $kx^3-lx^2+9kx-l=0\quad (k\cdot l\neq 0)$, then find the area of the triangle.

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by proposer

From the Cartan – Viete formula's, we have $a+b+c=\frac{l}{k}, ab+bc+ca=9, abc=\frac{l}{k}$. So

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=\frac{9}{\frac{l}{k}}=\frac{9k}{l}.$$ Now, we have $\frac{a+b+c}{3}=\frac{l}{3k}$, and

$\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{3}{\frac{9k}{l}}=\frac{l}{3k}$, namely $\frac{a+b+c}{3}=\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$. So, the result follows immediately from the fact that the AM-HM inequality becomes equality iff $a=b=c$. Namely the triangle is equilateral. From $ab+bc+ca=9$, we have $a^2=3$, and the area $=\frac{a^2\sqrt{3}}{4}=\frac{3\sqrt{3}}{4}$.

Solution 2 by Martin Celli-Mexico

For $x=a,b,c$ we have

$$kx(x^2+9)=l(x^2+1),$$

$$F(x)=\ln\left(\frac{l}{k}\right), \text{ where } F(x)=\ln\left(\frac{x(x^2+9)}{x^2+1}\right)\quad (x>0).$$

We can easily obtain the following expression:

$$F'(x)=\frac{(x^2-3)^2}{x(x^2+9)(x^2+1)}>0 \text{ for } x\neq\sqrt{3}.$$

Thus, the function F is strictly increasing. As $F(a)=F(b)=F(c)$, we have $a=b=c$: the triangle is equilateral, its area is $a^2\sqrt{3}/4$. On the other hand, we have

$$x^3-3ax^2+3a^2x-a^3=(x-a)^3=(x-a)(x-b)(x-c)=x^3-\frac{l}{k}x^2+9x-\frac{l}{k}$$

So $3a^2=9$, the area of the triangle is $3\sqrt{3}/4$.

Solution 3 by Marin Chirciu – Romania

By Viete relationships we have $a+b+c=\frac{l}{k}, ab+bc+ca=9, abc=\frac{l}{k}$

Using the identity in triangle $16S^2=2\sum b^2c^2-\sum a^4$.

We have $\sum b^2c^2 = 81 - \frac{2l^2}{k^2}$ and $\sum a^4 = \frac{l^4}{k^4} - \frac{32l^2}{k^2} + 162$ which follows from:

$$\sum b^2c^2 = \left(\sum bc\right)^2 - 2abc\sum a = 9^2 - 2\cdot\frac{l}{k}\cdot\frac{l}{k} = 81 - \frac{2l^2}{k^2}$$

$$(a+b+c)^4 = \sum a^4 + 6\sum b^2c^2 + 4\sum bc\,(b^2+c^2) + 12abc\sum a$$

$$\sum bc(b^2+c^2) = \sum a^2\sum bc - abc\sum a,$$

$$\sum bc(b^2+c^2) = \left(\frac{l^2}{k^2} - 18\right)\cdot 9 - \frac{l^2}{k^2} = \frac{8l^2}{k^2} - 162$$

$$\sum a^2 = \left(\sum a\right)^2 - 2\sum bc = \frac{l^2}{k^2} - 2\cdot 9 = \frac{l^2}{k^2} - 18$$

$$\frac{l^4}{k^4} = \sum a^4 + 6\left(81 - \frac{2l^2}{k^2}\right) + 4\left(\frac{8l^2}{k^2} - 162\right) + \frac{12l^2}{k^2} =$$

$$= \sum a^4 + 486 - \frac{12l^2}{k^2} + \frac{32l^2}{k^2} - 648 + \frac{12l^2}{k^2} =$$

$$= \frac{32l^2}{k^2} - 162 \Rightarrow \sum a^4 = \frac{l^4}{k^4} - \frac{32l^2}{k^2} + 162$$

We obtain $16S^2 = 2\sum b^2c^2 - \sum a^4 =$

$$= 2\left(81 - \frac{2l^2}{k^2}\right) - \left(\frac{l^4}{k^4} - \frac{32l^2}{k^2} + 162\right) = \frac{28l^2}{k^2} - \frac{l^4}{k^3} \Rightarrow$$

$$\Rightarrow 16S^2 = \frac{28l^2}{k^2} - \frac{l^4}{k^4} \Leftrightarrow S^2 = \frac{1}{16}\left(\frac{28l^2}{k^2} - \frac{l^4}{k^4}\right) \Leftrightarrow S = \frac{1}{4}\sqrt{\frac{28l^2}{k^2} - \frac{l^4}{k^4}}$$

It is necessary the condition $\frac{28l^2}{k^2} - \frac{l^4}{k^4} > 0 \Leftrightarrow \frac{1}{k} < 2\sqrt{7}$