SP.535 Determine all the numbers $\overline{abcd}$ such that:

$$1+a+b+c+a\cdot b+b\cdot c+c\cdot a=a\cdot b\cdot c\cdot d$$

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Solution 1 by proposers, Solution 2 by Marin Chirciu – Romania

Solution 1 by proposers

$$dabc-ab-bc-ac-a-b=c+1\Leftrightarrow (dc-1)ab-(c+1)a-(c+1)b=c+1$$
(1)

If $dc=1$, then $d=c=1$ so (1) doesn't solutions. Multiply (1) with $dc-1$, then

$$(dc-1)^2ab-(dc-1)(c+1)a-(dc-1)(c+1)b=(dc-1)(c+1)\Leftrightarrow$$
$$\Leftrightarrow [(dc-1)a-(c+1)][(dc-1)b-(c+1)]=(c+1)^2+(dc-1)(c+1)\Leftrightarrow$$
$$\Leftrightarrow [(dc-1)a-(c+1)][(dc-1)b-(c+1)]=(d+1)c(c+1)\quad (*)$$

WLOG we can assume that $a\ge b\ge c$. Then

$$(dc-1)a-(c+1)\ge dc^2-2c-1,(dc-1)b-(c+1)\ge dc^2-2c-1$$

We have the cases:

1. If $d\ge 3$, then $dc^2-2c-1\ge 3c^2-2c-1=(c-1)(3c+1)\ge 0$ and by (*) yields

$$(d+1)c(c+1)=[(dc-1)a-(c+1)][(dc-1)a-(c+1)]\ge (dc^2-2c-1)^2\Leftrightarrow$$
$$\Leftrightarrow d^2c^4+4c^2+1-4dc^3-2dc^2+4c-(d+1)c^2-(d+1)c\le 0\Leftrightarrow$$
$$\Leftrightarrow d^2c^4-4dc^3-3(d-1)c^2-(d-1)c+1\le 0\quad (2)$$

If $d\ge 8$, then

$$d^2c^4-4dc^3-3(d-1)c^2-(d-3)c+1\ge 8dc^4-4dc^3-3dc^2-dc+3c^2+3c+1=$$
$=4dc^3(c-1)+3dc^2(c^2-1)+dc(c^3-1)+3c^2+3c+1>0$ so (2) is false..

1.1 $d=7P$: (2) becomes $49c^4-28c^3-18c^2-4c+1\le 0\Leftrightarrow$

$\Leftrightarrow (c-1)(49c^3+21c^2+3c-1)\le 0$ and because

$49c^3+21c^2+3c-1>0$, we have $c=1$; (*) becomes

$(6a-2)(6b-2)+16\Leftrightarrow (3a-1)(3b-1)=4$ and we obtain $a=b=c=1$.

1.2. $d=6$: (2) becomes $36c^4-24c^3-15c^2-3c+1\le 0$

For $c\ge 2$: $36c^4-24c^3-15c^2-3c+1\ge 72c^3-24c^3-15c^2-3c+1=$

$=48c^3-15c^2-3c+1=30c^3+15c^2(c-1)+3c(c^2-1)+1>0$ and results

$c=1$; (*) becomes $(5a-2)(5b-2)=12$ and we not obtain solutions.

1.3. $d=5$: (2) becomes $25c^4-20c^3-12c^2-2c+1\le 0$

For $c\ge 2$ we have $25c^4-20c^3-12c^2-2c+1\ge 50c^3-20c^3-12c^2-2c+1=$

$=30c^3-12c^2-2c+1=16c^3+12c^2(c-1)+2c(c^2-1)+1>0$, so

$c=1$; (*) becomes

$(4a-2)(4b-2)=12\Leftrightarrow (2a-1)(2b-1)=3$ and we obtain the solution

$a=2,b=1,c=1$

1.4. $d=4$: (2) becomes $16c^4-16c^3-9c^2-c+1\le 0$. For $c\ge 2$ we have

$$16c^4-16c^3-9c^2-c+1\le 32c^3-16c^3-9c^2-c+1=$$

$=16c^3-9c^2-c+1=6c^3+9c^2(c-1)+c(c^2-1)+1>0$ and $c=1$; (*) becomes

$(3a-2)(3b-2)=10$ and we obtain solution $a=4,b=1,c=1$.

1.5 $d=3$; (2) becomes $9c^4-12c^3-6c^2+1\le 0$. For $c\ge 2$ we have

$9c^4-12c^3-6c^2+1\geq 18c^3-12c^3-6c^2+1=6c^3-6c^2+1=6c^2(c-1)+1>0$ so $c=1$; (*) becomes $(2a-2)(2b-2)=8\Leftrightarrow(a-1)(b-1)=2$ and we obtain $a=3,b=2,c+1$.

2. $d=2$

2.1. For $c=1$, (*) is $(a-2)(b-2)=6$ and we obtain the solutions $a=8,b=3,c=1$ and $a=5,b=4,c=1$.

2.2. For $c\geq 2$, $2c^2-2c-1\geq 4c-2c-1=2c-1>0$ and (2) is true, i.e.

$4c^4-8c^3-6c^2+c+1\leq 0$. But $c\geq 3$ so,

$4c^4-8c^3-6c^2+c+1\geq 12c^3-8c^3-6c^2+c+1=4c^3-6c^2+c+1=$

$=2c^3+2c^2(c-3)+c+1>0$, so $c=2$; (*) becomes

$(3a-3)(3b-3)=18\Leftrightarrow(a-1)(b-1)=2$, hence $a=3,b=2,c=2$.

3. $d=1$. Since $c\neq 1$, we have:

3.1. For $c=2$, (*) becomes $(a-3)(b-3)=12$ and we obtain

$(a=15,b=4,c=2),(a=9,b=5,c=2)$ and $(a=7,b=6,c=2)$.

3.2 For $c=3$, (*) becomes $(2a-4)(2b-4)=24\Leftrightarrow(a-2)(b-2)=6$ and we obtain $(a=8,b=3,c=3)$ and $(a=5,b=4,c=3)$.

3.3. For $c\geq 4$ we have $c^2-2c-1\geq 4c-2c-1=2c-1>0$ so (2), i.e.

$c^4-4c^3+2c+1\leq 0$, true

$c^4-4c^3+2c+1=c^4(c-4)+2c+1>0$, we not obtain solutions.

In conclusion,

a	b	c	d
1	1	1	7
2	1	1	5
4	1	1	4
3	2	1	2
5	4	1	2
8	3	1	2
3	2	2	2
15	4	2	1
9	5	2	1
7	6	2	1
8	3	3	1
5	4	3	1

Hence $\overline{abcd}\in\{1117,2115,3213,3222,4114,5412,5431,7621,8312,8331,9521\}$

Solution 2 by Marin Chirciu – Romania

Necessarily a,b,c,d are non-zero digits.

For $a=1\Rightarrow 2+2b+2c+bc=bcd\Leftrightarrow 2(1+b+c)=bc(d-1)$;

For $b=1$ agreed $c=1,2,4\Rightarrow d=7,5,4$. It follows $\overline{abcd}=1117,1125,1144$.

For $b=2$ agreed $c=1,3\Rightarrow d=5,3$. It follows $\overline{abcd}=1215,1233$.

For $b=3$ agreed $c=2,8\Rightarrow d=3,2$. It follows $\overline{abcd}=1323,1382$.

For $b=4$ agreed $c=1,5\Rightarrow d=4,2$. It follows $\overline{abcd}=1414,1452$.

For $b=5$ agreed $c=4\Rightarrow d=2$. It follows $\overline{abcd}=1542$.
For $b=6,7$ we don't have solutions.
For $b=8$ agreed $c=3\Rightarrow d=2$. It follows $\overline{abcd}=1832$.
For $b=9$ we don't have solutions.
For $a=2\Rightarrow 3+3b+3c+bc=2bcd\Leftrightarrow 3(1+b+c)=bc(2d-1)$;
For $b=1$ agreed $c=1,3\Rightarrow d=5,3$. It follows $\overline{abcd}=2115,2133$.
For $b=2$ agreed $c=3\Rightarrow d=2$. It follows $\overline{abcd}=2232$.
For $b=3$ agreed $c=1,2\Rightarrow d=3,2$. It follows $\overline{abcd}=2313,2322$.
For $b=4$ we don't have solutions.
For $b=5$ agreed $c=9\Rightarrow d=1$. It follows $\overline{abcd}=2591$.
For $b=6$ agreed $c=7\Rightarrow d=1$. It follows $\overline{abcd}=2671$.
For $b=7,8$ we don't have solutions.
For $b=9$ agreed $c=5\Rightarrow d=2$. It follows $\overline{abcd}=2951$.
For $a=3\Rightarrow 4+4b+4c+bc=3bcd\Leftrightarrow 4(1+b+c)=bc(3d-1)$;
For $b=1$ agreed $c=2,8\Rightarrow d=3,2$. It follows $\overline{abcd}=3123,3182$.
For $b=2$ agreed $c=1,2\Rightarrow d=3,2$. It follows $\overline{abcd}=3213,3222$.
For $b=3$ agreed $c=8\Rightarrow d=1$. It follows $\overline{abcd}=3381$.
For $b=4$ agreed $c=5\Rightarrow d=1$. It follows $\overline{abcd}=3451$.
For $b=5$ agreed $c=4\Rightarrow d=1$. It follows $\overline{abcd}=3541$.
For $b=6,7,8,9$ we don't have solutions.
For $a=4\Rightarrow 5+5b+5c+bc=4bcd\Leftrightarrow 5(1+b+c)=bc(4d-1)$;
For $b=1$ agreed $c=1,5\Rightarrow d=4,2$. It follows $\overline{abcd}=4114,4152$.
For $b=2$ we don't have solutions.
For $b=3$ agreed $c=5\Rightarrow d=1$. It follows $\overline{abcd}=4351$.
For $b=4$ we don't have solutions.
For $b=5$ agreed $c=1,3\Rightarrow d=2,1$. It follows $\overline{abcd}=4512,4531$.
For $b=6,7,8,9$ we don't have solutions.
For $a=5\Rightarrow 6+6b+6c+bc=5bcd\Leftrightarrow 6(1+b+c)=bc(5d-1)$;
For $b=1$ agreed $c=4\Rightarrow d=2$. It follows $\overline{abcd}=5142$.
For $b=2,3$ we don't have solutions.
For $b=4$ agreed $c=1,3\Rightarrow d=2,1$. It follows $\overline{abcd}=5412,5431$.
For $b=5,6,7,8,9$ we don't have solutions.
For $a=6\Rightarrow 7+7b+7c+bc=6bcd\Leftrightarrow 7(1+b+c)=bc(6d-1)$;
For $b=1$ we don't have solutions.
For $b=2$ agreed $c=7\Rightarrow d=1$. It follows $\overline{abcd}=6271$.
For $b=3,4,5,6$ we don't have solutions.
For $b=7$ agreed $c=2\Rightarrow d=1$. It follows $\overline{abcd}=6721$.
For $b=8,9$ we don't have solutions.
For $a=7\Rightarrow 8+8b+8c+bc=7bcd\Leftrightarrow 8(1+b+c)=bc(7d-1)$;
For $b=1$ we don't have solutions.
For $b=2$ agreed $c=6\Rightarrow d=1$. It follows $\overline{abcd}=7261$.

For $b=3,4,5$ we don't have solutions.
For $b=6$ agreed $c=2\Rightarrow d=1$. It follows $\overline{abcd}=7621$.
For $b=7,8,9$ we don't have solutions.
For $a=8\Rightarrow 9+9b+9c+bc=8bcd\Leftrightarrow 9(1+b+c)=bc(8d-1)$;
For $b=1$ agreed $c=3\Rightarrow d=2$. It follows $\overline{abcd}=8132$.
For $b=2$ we don't have solutions.
For $b=3$ agreed $c=1,3\Rightarrow d=2,1$. It follows $\overline{abcd}=8312,8331$.
For $b=4,5,6,7,8,9$ we don't have solutions.
For $a=9\Rightarrow 10+10b+10c+bc=9bcd\Leftrightarrow 10(1+b+c)=bc(9d-1)$;
For $b=1$ we don't have solutions.
For $b=2$ agreed $c=5\Rightarrow d=1$. It follows $\overline{abcd}=9251$.
For $b=3,4$ we don't have solutions.
For $b=5$ agreed $c=2\Rightarrow d=1$. It follows $\overline{abcd}=9521$.
For $b=6,7,8,9$ we don't have solutions.
Finally, we deduce:

$$\overline{abcd}=1117,1125,1144,1215,1233,1223,1382,1414,1452,1542,1832,2115,2133,2232,2313,2322,2591,2671,2951,3123,3182,3213,3222,3381,3151,3541,4114,4152,4351,4512,4531,5142,5412,5431,6271,6721,7261,7621,8132,8312,8331,9251,9251$$