ROMANIAN MATHEMATICAL MAGAZINE

If $x \geq 0$ then:

$$\frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2}\,dt\right)^2+\int_0^{2x} e^{-t^2}\,dt \geq 2\int_0^x e^{-t^2}\,dt$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ahmed Salem-Tunisia

Let $Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{v^2}{2}}\,dv$

Prob. that standard Gaussian RVX exceeds x

$$\int_0^x e^{-t^2}\,dt \overset{u=\sqrt{2}t}{=} \sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}x} e^{-\frac{u^2}{2}}\frac{1}{\sqrt{2}}du=\sqrt{\pi}\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}x} e^{-\frac{u^2}{2}}du$$

$$=\sqrt{\pi}\left(\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}du-\frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}x}^\infty e^{-\frac{u^2}{2}}du\right)=\sqrt{\pi}\left(\frac{1}{2}-Q(\sqrt{2}x)\right)$$

The given inequality is equivalent to

$$\frac{2}{\sqrt{\pi}}\cdot\pi\left(\frac{1}{2}-Q(\sqrt{2}x)\right)^2+\sqrt{\pi}\left(\frac{1}{2}-Q(2\sqrt{2}x)\right)\geq 2\sqrt{\pi}\left(\frac{1}{2}-Q(\sqrt{2}x)\right)$$

$$\Rightarrow \frac{\sqrt{\pi}}{2}+2\sqrt{\pi}Q^2(\sqrt{2}x)-2\sqrt{\pi}Q(\sqrt{2}x)+\frac{\sqrt{\pi}}{2}-\sqrt{\pi}Q(2\sqrt{2}x)\geq\sqrt{\pi}-2\sqrt{\pi}Q(\sqrt{2}x)$$

$$\Rightarrow 2Q^2(\sqrt{2}x)\geq Q(2\sqrt{2}x)$$

$$Q(x)=\frac{1}{2\pi}\int_x^\infty\int_{-\infty}^\infty e^{-\frac{1}{2}(u^2+v^2)}\,dudv=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_{x\sec\phi}^\infty -d\,e^{-\frac{1}{2}r^2}\,d\phi=$$

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}x^2\sec^2\phi}\,d\phi \overset{\phi\to\frac{\pi}{2}-\phi}{=} \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\phi}}\,d\phi$$

$$\left(1-2Q(x)\right)^2=\frac{1}{2\pi}\int_{-x}^x\int_x^x e^{-\frac{1}{2}(u^2+v^2)}\,du\,dv=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\int_0^{x\sec\phi} -de^{-\frac{1}{2}r^2}\,d\phi$$

$$=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\left(1-e^{-\frac{1}{2}x^2\sec^2\phi}\right)d\phi=1-4Q(x)+4Q^2(x)$$

$$\Rightarrow Q^2(x)=Q(x)-\frac{1}{\pi}\int_0^{\frac{\pi}{4}} e^{-\frac{1}{2}x^2\sec^2\phi}=\frac{1}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-\frac{1}{2}x^2\sec^2\phi}\,d\phi \overset{\phi\to\frac{\pi}{2}-\phi}{=} \frac{1}{2}\int_0^{\frac{\pi}{4}} e^{-\frac{x^2}{2\sin^2\phi}}\,d\phi$$

$$2Q^2(\sqrt{2}x)=\frac{2}{\pi}\int_0^{\frac{\pi}{4}} e^{-\frac{x^2}{\sin\phi}}\,d\phi$$

$$Q(2\sqrt{2}x)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{4x^2}{\sin B}}\,dB \overset{\phi=\frac{B}{2}}{=} \frac{2}{\pi}\int_0^{\frac{\pi}{4}} e^{-\frac{4x^2}{\sin^2(2\phi)}}\,d\phi$$

$$2Q^2(\sqrt{2}x)-Q(2\sqrt{2}x)=\frac{2}{\pi}\int_0^{\frac{\pi}{4}}\left(e^{-\frac{x^2}{\sin^2\phi}}-e^{-\frac{4x^2}{\sin^2(2\phi)}}\right)d\phi$$

For $\phi\in\left(0,\frac{\pi}{4}\right]$:

$$4\sin^2\phi>4\sin^2\phi\cos^2\phi=\sin^2 2\phi\Rightarrow\frac{4}{\sin^2 2\phi}>\frac{1}{\sin^2\phi}\Rightarrow-\frac{x^2}{\sin^2\phi}>-\frac{4x^2}{\sin^2 2\phi}\Rightarrow$$

$$\Rightarrow e^{-\frac{x^2}{\sin^2\phi}}>e^{-\frac{4x^2}{\sin^2(2\phi)}}$$

$$\Rightarrow 2Q^2(\sqrt{2}x)\ge Q(2\sqrt{2}x),\frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2}dt\right)^2+\int_0^{2x} e-t^2\,dt\ge 2\int_0^x e^{-t^2}\,dt$$

Solution 2 by Djamel Arrouche-France

$$\frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2}dt\right)^2+\int_0^{2x} e^{-t^2}\,dt\ge 2\int_0^x e^{-t^2}\ \dots\ \text{E}$$

$$\text{erf}(y)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

$$E\Leftrightarrow \text{erf}^2(x)+\text{erf}(2x)\ge 2\,\text{erf}(x)$$

Proposition $\forall(x,y)\in[0,\infty[^2$

$$\text{erf}(x)\,\text{erf}(y)+\text{erf}(x+y)\ge\text{erf}(x)+\text{erf}(y)$$

Let $y\in\mathbb{R}_+\ \forall x\ge 0$

$$f_y(x)=\text{erf}(x)\,\text{erf}(y)+\text{erf}(x+y)-\text{erf}(x)-\text{erf}(y)$$

$$\Rightarrow f_y'(x)=\frac{2}{\sqrt{\pi}}\left(e^{-x^2}\,\text{erf}(y)+e^{-(x+y)^2}-e^{-x^2}\right)$$

$$\Leftrightarrow\frac{\sqrt{\pi}}{2}e^{-x^2}f'(x)=\text{erf}(y)+e^{-y^2-2xy}-1$$

$$f_y'(x)\ge 0\Leftrightarrow e^{-y^2-2xy}\ge 1-\text{erf}(y)\Rightarrow x\le\frac{1}{-2y}\left(\ln(1-\text{erf}(y))+y\right)=a_y$$

$$1-\text{erf}(y)=\frac{2}{\sqrt{\pi}}\int_y^\infty e^{-t^2}\,dt;\,t=y+z\Rightarrow\frac{2}{\sqrt{\pi}}e^{-y^2}\int_0^\infty e^{-z^{2-yz}}\,dz\le e^{-y^2}$$

$$a_y \geq \frac{1}{-2y}(-2x+y) = \frac{1}{2} \Rightarrow a_y \in [0,+\infty[$$

$$f_y(x) \leq f(a_y)$$

$$\Rightarrow f_y(x) = \min\left(f_y(0), \lim_{x\to\infty} f_y(x)\right) \Rightarrow f_y(0) = 0$$

$$\lim_{y\to\infty} f_y(x) = \text{erf}(y) + 1 - 1 - \text{erf}(y) = 0$$

$$f_y(x) \geq 0 \Rightarrow f_y(x) \geq 0; \forall x \in [0,\infty[$$

$\Rightarrow$ **since** y **is arbitrary** $\Rightarrow \forall (x,y) \in \mathbb{R}_+^2$

$$\text{erf}(x)\,\text{erf}(y) + \text{erf}(x+y) \geq \text{erf}(x) + \text{erf}(y)$$

$$c = y \Rightarrow \text{erf}^2(x) + \text{erf}(2x) \geq 2\,\text{erf}(x)$$