ROMANIAN MATHEMATICAL MAGAZINE

SP.537 Solve for real numbers:

$$3e^x + 3e^{3x} + 1 = 4e^{2x} + 3 \cdot \sqrt[3]{e^{4x}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$3e^x + 3e^{3x} + 1 = (2e^x + 2e^{3x}) + (e^x + e^{3x} + 1) =$$

$$= 2(e^x + e^{3x}) + (e^x + e^{3x} + 1) \overset{AM-GM}{\geq} 2 \cdot 2\sqrt{e^x \cdot e^{3x}} + 3 \cdot \sqrt[3]{e^x \cdot e^{3x} \cdot 1} =$$

$$= 4\sqrt{e^{4x}} + 3\sqrt[3]{e^{4x}} = 4e^{2x} + 3\sqrt[3]{e^{4x}}$$

Equality holds for: $e^x = e^{3x}$ and $e^x = e^{3x} = 1$. Solution: $x = 0$

Solution 2 by Marin Chiricu – Romania

We have $3e^x + 3e^{3x} + 1 = 4e^{2x} + 3\sqrt[3]{e^{4x}} \Leftrightarrow 3e^x + 3e^{3x} + 1 = 4e^{2x} + 3e^{\frac{4x}{3}}$

With the substitution $e^{\frac{x}{3}} = t > 0$ the equation becomes

$$3t^3 + 3t^9 + 1 = 4t^6 + 3t^4 \Leftrightarrow 3t^9 - 4t^6 - 3t^4 + 3t^3 + 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow (t-1)^2(3t^7 + 6t^6 + 9t^5 + 8t^4 + 7t^3 + 3t^2 + 2t + 1) = 0 \Leftrightarrow t = 1.$$

Returning to the notations it follows $e^{\frac{x}{3}} = 1 \Leftrightarrow x = 0$

Reciprocal $x = 0$ verify the equation.

We deduce that $x = 0$ is the unique solution of the equation.