ROMANIAN MATHEMATICAL MAGAZINE

SP.538 In acute ΔABC the following relationship holds:

$$36 \leq 4\left(\sum_{cyc} \tan A \tan B\right) \leq 9 + \prod_{cyc} \tan^2 A$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer, Solution 2 by Marin Chirciu – Romania

Solution 1 by proposer

$$\frac{1}{3}\sum_{cyc} \tan A \overset{AM-HM}{\geq} \frac{3}{\sum_{cyc} \frac{1}{\tan A}}$$

$$\left(\sum_{cyc} \tan A\right) \cdot \left(\sum_{cyc} \frac{1}{\tan A}\right) \geq 9$$

$$\prod_{cyc} \tan A \cdot \frac{\tan A \tan B + \tan B \tan C + \tan C \tan A}{\prod_{cyc} \tan A} \geq 9$$

$$\sum_{cyc} \tan A \tan B \geq 9 \Rightarrow 4\sum_{cyc} \tan A \tan B \geq 36$$

By Padoa's inequality:

$$\prod_{cyc}(\tan A + \tan B - \tan C) \leq \prod_{cyc} \tan A \quad (1)$$

Denote: $u = \sum_{cyc} \tan A = \prod_{cyc} \tan A > 0$

By (1):

$$\prod_{cyc}(u - \tan A) \leq u$$

$$u^3 - 2\left(\sum_{cyc} \tan A\right) \cdot u^2 + 4\left(\sum_{cyc} \tan A \tan B\right) \cdot u - 8u \leq u$$

$$u^3 - 2u^3 + 4\left(\sum_{cyc} \tan A \tan B\right) \cdot u \leq 9u$$

$$4\left(\sum_{cyc} \tan A \tan B\right) \cdot u \leq 9u + u^3$$

$$4\sum_{cyc}\tan A\tan B\le u^2+9$$

$$4\sum_{cyc}\tan A\tan B\le 9+\prod_{cyc}\tan^2 A$$

Equality holds for:

$$\tan A=\tan B=\tan C\Rightarrow A=B=C=\frac{\pi}{3}$$

Solution 2 by Marin Chirciu – Romania

Changing the variable $A\to\frac{\pi}{2}-A$, the inequality from enunciation is equivalent with:

$$36\le 4\sum\cot\frac{A}{2}\cot\frac{B}{2}\le 9+\prod\cot^2\frac{A}{2}$$

Using the inequalities in triangle $\sum\cot\frac{A}{2}\cot\frac{B}{2}=\frac{4R+r}{r}$ and $\prod\cot\frac{A}{2}=\frac{s}{r}$

First inequality.

$$36\le 4\sum\cot\frac{A}{2}\cot\frac{B}{2}\Leftrightarrow\sum\cot\frac{A}{2}\cot\frac{B}{2}\ge 9\Leftrightarrow\frac{4R+r}{r}\ge 9\Leftrightarrow R\ge 2r, \text{(Euler)}$$

Equality holds if and only if the triangle is equilateral.

Second inequality.

$$4\sum\cot\frac{A}{2}\cot\frac{B}{2}\le 9+\prod\cot^2\frac{A}{2}\Leftrightarrow 4\cdot\frac{4R+r}{r}\ge 9+\frac{s^2}{r^2}\Leftrightarrow s^2\ge 16Rr-5r^2, \text{(Gerretsen)}$$

Equality holds if and only if the triangle is equilateral.