ROMANIAN MATHEMATICAL MAGAZINE

SP.539. If $a>0$; $f:\mathbb{R}\to\mathbb{R}$ is a function such that:

$$f\left(ax-\frac{1}{a}\right)\le ax\le f(x)-1;(\forall)x\in\mathbb{R}\ \text{ then:}$$

$$f(2)+f(4)+f(8)>\frac{12\sqrt{a}}{a}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$ax\le f(x)-1\Rightarrow f(x)\ge\frac{ax+1}{a}\quad (1)$$

$$f\left(ax-\frac{1}{a}\right)\le ax\quad (2)$$

Replace $y=ax-\frac{1}{a}$ in (2): $y+\frac{1}{a}=ax\Rightarrow x=\frac{1}{a}y+\frac{1}{a^2}$

$$f(y)\le a\left(\frac{1}{a}y+\frac{1}{a^2}\right)\Rightarrow f(y)\le y+\frac{1}{a}=\frac{ay+1}{a}$$

$$f(x)\le\frac{ax+1}{a}\quad (3)$$

By (2);(3):

$$f(x)=\frac{ax+1}{a}\Rightarrow f(x)=x+\frac{1}{a}$$

$$f(2)+f(4)+f(8)=\left(2+\frac{1}{a}\right)+\left(4+\frac{1}{a}\right)+\left(8+\frac{1}{a}\right)\ge$$

$$\overset{AM-GM}{>}6\cdot\sqrt[6]{2\cdot\frac{1}{a}\cdot4\cdot\frac{1}{a}\cdot8\cdot\frac{1}{a}}=6\sqrt[6]{2^6\cdot\frac{1}{a^3}}=12\cdot\sqrt[6]{\frac{1}{a^3}}=12\cdot\frac{1}{\sqrt{a}}=\frac{12\sqrt{a}}{a}$$

Solution 2 by Marin Chirciu – Romania

We have $f(x)\ge ax+1,(\forall)x\in\mathbb{R}$.

We obtain $f(2)+f(4)+f(8)\ge(2a+1)+(4a+1)+(8a+1)=14a+3>\frac{12\sqrt{a}}{a}$

which follows from $14a+3>\frac{12\sqrt{a}}{a},a>0$

We denote $\sqrt{a}=t>0$ and $14a+3>\frac{12\sqrt{a}}{a}$ we write $14t^2+3>\frac{12}{t}\Leftrightarrow$

$\Leftrightarrow 14t^3+3t+12>0,t>0$

We consider the function $g(t)=14t^3+3t-12,t>0$

$g'(t)=42t^2+3>0\Rightarrow g$ is strictly increasing on $(0,\infty)$.

From the table of variation it follows $g(t)>0$ for $t\ge1$.