ROMANIAN MATHEMATICAL MAGAZINE

SP.540 If $x, y \in [3,4]; z, t \in [1,12]$ then:

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right) \le \frac{100}{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$x \in [3,4] \Rightarrow 3 \le x \le 4 \Rightarrow x-3 \ge 0; x-4 \le 0$$

$$\Rightarrow (x-3)(x-4) \le 0 \Rightarrow x^2-7x+12 \le 0$$

$$\Rightarrow x^2+12 \le 7x \Rightarrow x+\frac{12}{x} \le 7 \Rightarrow 7 \ge x+\frac{12}{x} \quad (1)$$

Analogous: $y \in [3,5] \Rightarrow 7 \ge y+\frac{12}{y}$ (2)

$$z \in [1,12] \Rightarrow 1 \le z \le 12 \Rightarrow z-1 \ge 0; z-12 \le 0$$

$$\Rightarrow (z-1)(z-12) \le 0 \Rightarrow z^2-13z+12 \le 0$$

$$\Rightarrow z^2+12 \le 13z \Rightarrow z+\frac{12}{z} \le 13 \Rightarrow 13 \ge z+\frac{12}{z} \quad (3)$$

Analogous: $t \in [1,12] \Rightarrow 13 \ge t+\frac{12}{t}$ (4)

By adding (1);(2);(3);(4):

$$40 \ge x+y+z+t+12\left(\frac{1}{x}+\frac{1}{t}+\frac{1}{z}+\frac{1}{t}\right) \ge$$

$$\overset{AM-GM}{\ge} 2\sqrt{(x+y+z+t)\cdot 12\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)} =$$

$$= 4\sqrt{3(x+y+z+t)\cdot 3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}$$

$$10 \ge \sqrt{3(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)}$$

$$100 \ge 3(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)$$

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right) \le \frac{100}{3}$$

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Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\le\left(\frac{x+y+z+t}{4\sqrt{3}}+\sqrt{3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\right)^2$$
$$=\left[\sum_{x,y,z,t}\left(\frac{x}{4\sqrt{3}}+\frac{\sqrt{3}}{x}\right)\right]^2.$$

We have $(x-3)(x-4)\le 0$, **then** $x^2+12\le 7x$ **or** $\frac{x}{4\sqrt{3}}+\frac{\sqrt{3}}{x}\le\frac{7}{4\sqrt{3}}$.

Similarly, we have $\frac{y}{4\sqrt{3}}+\frac{\sqrt{3}}{y}\le\frac{7}{4\sqrt{3}}$.

Also, we have $(z-1)(z-12)\le 0$, **then** $z^2+12\le 13z$ **or** $\frac{z}{4\sqrt{3}}+\frac{\sqrt{3}}{z}\le\frac{13}{4\sqrt{3}}$.

Similarly, we obtain $\frac{t}{4\sqrt{3}}+\frac{\sqrt{3}}{t}\le\frac{13}{4\sqrt{3}}$.

Therefore

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\le\left(2.\frac{7}{4\sqrt{3}}+2.\frac{13}{4\sqrt{3}}\right)^2=\frac{100}{3}.$$

Equality holds iff $x,y\in\{3,4\}$ **and** $z,t\in\{1,12\}$.