ROMANIAN MATHEMATICAL MAGAZINE

UP.526. Prove the identity:

$$\int_0^{\infty} \frac{|\sin(x)|}{1+x^2} dx = 1 - 2\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)e^{2n}}$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote:

$$I = \int_0^{\infty} \frac{|\sin(x)|}{1+x^2} dx$$

The function $f(x) = |\sin(x)|$ is periodic with period π and satisfies Dirichlet's conditions. Also, the function is even. We expand the function in the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx)$$

where:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\, dx; a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(2nx)\, dx$$

Calculating these integrals, we obtain:

$$a_0 = \frac{2}{\pi}; a_n = -\frac{4}{\pi}\cdot\frac{1}{4n^2-1}$$

We have:

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2-1}$$

Substituting $f(x)$ in the expression of I, result:

$$I = \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+x^2} dx - \frac{4}{\pi}\int_0^{\infty} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{(4n^2-1)(1+x^2)}$$

So:

$$I = 1 - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\int_0^{\infty} \frac{\cos(2nx)}{1+x^2} dx$$

We now use the following relationship:

$$\int_0^{\infty} \frac{\cos(mx)}{1+x^2} dx = \frac{\pi}{2} e^{-m}, m > 0$$

This relation is Laplace's integral and is well known. It is easily proved for example using the properties of the Laplace transform. We obtained the value of the integral I:

$$I = 1 - 2\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)e^{2n}}$$

Solution 2 by Pham Duc Nam-Vietnam

*** The function $f(x) = |\sin(x)|$ is even function then $b_n = 0$ and the Fourier series collapses to**

$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$ **and** $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$

$$x \in [0,\pi] \Rightarrow \sin(x) \geq 0 \Rightarrow |\sin(x)| = \sin(x)$$

Then the Fourier series of $|\sin(x)|$ is given by:

$$f(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx + \sum_{n=2}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(nx)\,dx\right)\cos(nx) \because a_1 = 0$$

1. $\frac{1}{\pi}\int_0^{\pi}\sin(x)\,dx = \frac{1}{\pi}(-\cos(x))|_0^{\pi} = \frac{2}{\pi}$

2. $\frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(\sin(x(1+n)) + \sin\big(x(1-n)\big)\right)dx =$

$$= -\frac{1}{\pi}\frac{2(\cos(\pi n)+1)}{n^2-1} = \begin{cases} 0, if\ n\ is\ odd \\ -\frac{1}{\pi}\frac{4}{n^2-1}, if\ n\ is\ even \end{cases}$$

$$\Rightarrow f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}$$

$$* \int_0^{\infty}\frac{|\sin(x)|}{1+x^2}dx = \int_0^{\infty}\frac{1}{1+x^2}\left(\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}\right)dx =$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}dx - \frac{4}{\pi}\sum_{k}^{\infty}\frac{1}{4k^2-1}\int_0^{\infty}\frac{\cos(2kx)}{1+x^2}dx$$

$$= 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\int_{-\infty}^{\infty}\frac{e^{2kix}}{1+x^2}dx\right) =$$

$$= 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\left(2\pi i. Res\left(\frac{e^{2kix}}{1+z^2}, z = i\right)\right)\right)$$

$$= 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\left(2\pi i \lim_{x\to i}(z-i)\frac{e^{2kix}}{(z+i)(z-i)}\right)\right) =$$

$$= 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\frac{1}{2}\mathcal{R}\left(2\pi i\left(-\frac{1}{2}ie^{-2k}\right)\right)\right)$$

$$= 1 - 2\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\frac{1}{e^{2k}}, \text{ as required.}$$

Solution 3 by Ankush Kumar Parcha-India

Consider, $f(z) := \frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)$ &$\Gamma: \gamma \cup (-r,-\delta) \cup \psi \cup (\delta, R)$

is the contour containing γ and ψ a semi-circular arcs in the upper half of $\mathbb{C}$ - plane with different radii and $(-R,-\delta) \cup (\delta, R)$ lies on $\mathcal{R}(z)$ axis

$$\oint_{\Gamma}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz = \int_{\gamma}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz + \int_{-R}^{-\delta}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$+\int_{\psi}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz + \int_{\delta}^{R}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

($\because$ Cauchy's Residue Theorem: $\int_C f(z)\,dz = 2\pi i\sum_k Res f(z = z_k)$)

($\because$ Jordan's Lemma: $\lim_{R\to\infty}\int_{C_R} e^{imz}f(z)\,dz = 0, f(z) \to 0 \wedge z \to \infty \wedge m \in \mathbb{R}^+$)

$$\overset{Taking\ limits}{\underset{R\to\infty,\delta\to 0}{\Longrightarrow}} \lim_{\substack{R\to\infty\\ \delta\to 0}} 2\pi i\, Res\, f(z=i) + \lim_{\substack{R\to\infty\\ \delta\to 0}} 2\pi i \sum_{n\in\mathbb{N}} Res\, f(z = 2ni) =$$

$$= \underbrace{\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\gamma}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz}_{=0,\because f(z)\to 0 \wedge z\to\infty\wedge 1\in\mathbb{R}^+}$$

$$+\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{-R}^{-\delta}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx + \underbrace{\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\psi}\frac{e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)dz}_{=-i\pi\, Res\, f(z=0)} +$$

$$+\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\delta}^{R}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$\Rightarrow 2\pi i\underbrace{\lim_{z\to i}\frac{(z-i)e^{iz}\coth\left(\frac{\pi z}{2}\right)}{1+z^2}}_{=0} + 2\pi i\sum_{n\in\mathbb{N}}\lim_{z\to 2ni}\frac{(z-2ni)e^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)$$

$$= \int_{\mathbb{R}^-}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx - i\pi\underbrace{\lim_{z\to 0}\frac{ze^{iz}}{1+z^2}\coth\left(\frac{\pi z}{2}\right)}_{=\frac{2}{\pi}} + \int_{\mathbb{R}^+}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$\overset{Taking\ Imaginary}{\underset{part}{\Longrightarrow}} \Im\left\{2\pi i\left(-\frac{2}{3\pi e^2} - \frac{2}{15\pi e^4} - \frac{2}{35\pi e^6} - \cdots\right)\right\} = \Im\{-2i\}$$

$$+\Im\int_{\mathbb{R}}\frac{e^{ix}}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx \Rightarrow 2-4\sum_{n\in\mathbb{N}}\frac{1}{(4n^2-1)e^{2n}}=\int_{\mathbb{R}}\frac{\sin(x)}{1+x^2}\coth\left(\frac{\pi x}{2}\right)dx$$

$$\left(\because \int_{-a}^{+a}f(t)\,dt=2\int_0^a f(t), f(-t)=f(t)\right)$$

$$\left(\because \textit{Maz Identity}: \int_{\mathbb{R}^+}f(t)\cdot g(t)dt=\int_{\mathbb{R}^+}\mathcal{L}_t\{f(t)\}(x)\cdot\mathcal{L}_t^{-1}\{g(t)\}(x)dx\right)$$

$$\Rightarrow 1-2\sum_{n\in\mathbb{N}}\frac{1}{(4n^2-1)e^{2n}}=\int_{R^+}\mathcal{L}_x\{\sin(x)\}(t)\cdot\mathcal{L}_x^{-1}\left\{\frac{\coth\left(\frac{\pi x}{2}\right)}{1+x^2}\right\}(t)dt$$

$$\left(\because \mathcal{L}_t\{|\sin(t)|\}(s)=\frac{\coth\left(\frac{\pi s}{2}\right)}{1+s^2}\right)$$

$$\left(\because \mathcal{L}_t\left\{\sin(\omega t)\,(s)=\frac{\omega}{s^2+\omega^2}\right\}, s>|\Im(\omega)|\right)$$

$$\overset{t-x}{\Longrightarrow}\int_{\mathbb{R}^+}\frac{|\sin(x)|}{1+x^2}dx=1-2\sum_{n\in\mathbb{N}}\frac{1}{(4n^2-1)e^{2n}}$$