UP.527 Prove the closed form:

$$\int_0^{\infty}\frac{\ln x}{x^3+x\sqrt{x}+1}dx=-\frac{32\pi^2}{81}\sin\frac{\pi}{18}$$

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Solution 1 by proposer

Let us denote:

$$I=\int_0^{\infty}\frac{\ln x}{x^3+x\sqrt{x}+1}dx;\ A=\int_0^{1}\frac{\ln x}{x^3+x\sqrt{x}+1}dx;\ B=\int_1^{\infty}\frac{\ln x}{x^3+x\sqrt{x}+1}dx$$

We consider the integral A. We make the variable change: $x=y^{\frac{2}{3}}$

We have, successively:

$$A=\frac{4}{9}\int_0^1\frac{(1-y)y^{-\frac{1}{3}}\ln y}{1-y^3}dy=\frac{4}{9}\left(\int_0^1\frac{y^{\frac{1}{3}}\ln y}{1-y^3}dy-\int_0^1\frac{y^{\frac{2}{3}}\ln y}{1-y^3}dy\right)$$

$$A=\frac{4}{9}\left(\int_0^1\sum_{k=0}^{\infty}y^{3k-\frac{1}{3}}\ln y\,dy-\int_0^1\sum_{k=0}^{\infty}y^{3k-\frac{2}{3}}\ln y\,dy\right);$$

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left(\int_0^1 y^{3k-\frac{1}{3}}\ln y\,dy-\int_0^1 y^{3k+\frac{2}{3}}\ln y\,dy\right)$$

We will use the following relationship:

$$\int_0^1 x^a\ln x\,dx=-\frac{1}{(a+1)^2},\text{ where } a\in\mathbb{R}, a\geq 0.$$

We obtain:

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left[\frac{1}{\left(3k+\frac{5}{3}\right)^2}-\frac{1}{\left(3k+\frac{2}{3}\right)^2}\right];\ A=\frac{4}{9}\sum_{k=0}^{\infty}\left[\frac{\frac{1}{9}}{\left(k+\frac{5}{9}\right)^2}-\frac{\frac{1}{9}}{\left(k+\frac{2}{9}\right)^2}\right]$$

We now use the following relationship:

$$\Psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\Psi_1(x)$ is the trigamma function.We obtained the value of the integral A:

$$A=\frac{4}{81}\left[\Psi_1\left(\frac{5}{9}\right)-\Psi_1\left(\frac{2}{9}\right)\right]$$

We consider the integral B. We make the variable change: $x=\frac{1}{y}$. Then, by proceeding to the integral A, we obtain:

$$B=\frac{4}{81}\left[\Psi_1\left(\frac{4}{9}\right)-\Psi_1\left(\frac{7}{9}\right)\right]$$

Result:

$$I=A+B=\frac{4}{81}\left[\Psi_1\left(\frac{5}{9}\right)-\Psi_1\left(\frac{2}{9}\right)+\Psi_1\left(\frac{4}{9}\right)-\Psi_1\left(\frac{7}{9}\right)\right]$$

We use the reflection formula:

$$\Psi_1(x)+\Psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\Psi_1\left(\frac{2}{9}\right)+\Psi_1\left(\frac{7}{9}\right)=\frac{\pi^2}{\sin^2\frac{2\pi}{9}};\ \Psi_1\left(\frac{4}{9}\right)+\Psi_1\left(\frac{5}{9}\right)=\frac{\pi^2}{\sin^2\frac{4\pi}{9}}$$

Result:

$$I=\frac{4\pi^2}{81}\left(-\frac{1}{\sin^2\frac{2\pi}{9}}+\frac{1}{\sin^2\frac{4\pi}{9}}\right)$$

We have:

$$-\frac{1}{\sin^2\frac{2\pi}{9}}+\frac{1}{\sin^2\frac{4\pi}{9}}=-8\sin\frac{\pi}{18}$$

We will prove this equality. We use the relationships

$$\sin 3a=\sin a\,(1+2\cos 2a);\cos 3a=\cos a\,(2\cos 2a-1)$$

We consider

$$E=\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\sin^2\frac{4\pi}{9}}=\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\cos^2\frac{\pi}{18}}=\frac{\left(1+2\cos\frac{4\pi}{9}\right)^2}{\sin^2\frac{2\pi}{3}}-\frac{\left(2\cos\frac{\pi}{9}-1\right)^2}{\cos^2\frac{\pi}{6}}$$

$$E=\frac{16}{3}\left(\cos\frac{4\pi}{9}+\cos\frac{\pi}{9}\right)\left(1+\cos\frac{4\pi}{9}-\cos\frac{\pi}{9}\right)=\frac{16\sqrt{3}}{3}\cos\frac{5\pi}{18}\left(1-\sin\frac{5\pi}{18}\right)$$

$$E=\frac{8\sqrt{3}}{3}\left(2\cos\frac{5\pi}{18}-\sin\frac{5\pi}{9}\right)$$

But: $\cos\frac{5\pi}{18}=\frac{1}{2}\cos\frac{\pi}{18}+\frac{\sqrt{3}}{2}\sin\frac{\pi}{18}$; $\sin\frac{5\pi}{9}=\cos\frac{\pi}{18}$. **So:** $E=8\sin\frac{\pi}{18}$

Result:

$$I=-\frac{32\pi^2}{81}\sin\frac{\pi}{18}$$

Solution 2 by Odeyemi Gideon-Nigeria

$$\int_0^{+\infty}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx=\int_0^{+\infty}\frac{\left(x^{\frac{3}{2}}-1\right)\ln(x)}{x^{\frac{9}{2}}-1}dx=$$

$$=\int_0^{+\infty}\left(\frac{x^{\frac{3}{2}}\ln(x)}{x^{\frac{9}{2}}-1}-\frac{\ln(x)}{x^{\frac{9}{2}}-1}\right)dx$$

$$\overset{x^{\frac{9}{2}}=x}{=}\frac{4}{81}\int_0^{+\infty}\left(\frac{x^{-\frac{4}{9}}\ln(x)}{x-1}-\frac{x^{-\frac{7}{9}}\ln(x)}{x-1}\right)dx$$

Let's consider $\int_0^{+\infty}\frac{x^a\ln(x)}{x-1}dx$

$$\int_0^{+\infty}\frac{x^a\ln(x)}{x-1}dx=\frac{d}{da}\left(\int_0^{+\infty}\frac{x^a}{x-1}dx\right)\overset{\frac{1}{x-1}=x}{=}\frac{d}{da}\left((-1)^a\int_0^1 x^{-a-1}(1-x)^a dx\right)$$

$$=\frac{d}{da}\left((-1)^a\Gamma(-a)\Gamma(a+1)\right)\overset{E.R.F}{=}\frac{d}{da}\left((-1)^a\frac{\pi}{\sin(-\pi a)}\right)=\frac{d}{da}\left((-1)^{a+1}\frac{\pi}{\sin(\pi a)}\right)$$

$$\Rightarrow\frac{4}{81}\int_0^{+\infty}\left(\frac{x^{-\frac{4}{9}}\ln(x)}{x-1}-\frac{x^{-\frac{7}{9}}\ln(x)}{x-1}\right)dx=\frac{4}{81}\left(\pi^2\csc^2(\pi a)\right)_{a=-\frac{7}{9}}^{x=-\frac{4}{9}}$$

$$=\frac{4\pi^2}{81}\left(\sec^2\left(\frac{\pi}{18}\right)-\csc^2\left(\frac{2\pi}{9}\right)\right)=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{18}\right)$$

Solution 3 by Pham Duc Nam-Vietnam

$$\Omega=\int_0^{\infty}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx=-\frac{32}{81}\pi^2\sin\left(\frac{\pi}{18}\right)?$$

$$\Omega=\int_0^{\infty}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx, t=x\sqrt{x}\Rightarrow dx=\frac{2}{3}\frac{1}{t^{\frac{1}{3}}}dt$$

$$\Rightarrow\Omega=\frac{4}{9}\int_0^{\infty}\frac{t^{-\frac{1}{3}}\ln(t)}{t^2+t+1}dt=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},x=\frac{1}{2}}\int_0^{\infty}\frac{t^{s-1}}{t^2+2xt+1}dt=$$

$$=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},x=\frac{1}{2}}\int_0^\infty t^{s-1}\left(\sum_{n=0}^{\infty}U_n(x)(-t)^n\right)dt$$

Where: $U_n(x)$ is the Chebyshev polynomials of the second kind, and its generating function is:

$$\sum_{n=0}^{\infty}U_n(x)\,t^n=\frac{1}{t^2-2xt+1},\text{ letting } x=\cos(\theta)\Rightarrow U_n(x)=\frac{\sin((n+1)\theta)}{\sin(\theta)}$$

$$\Rightarrow\Omega=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},\theta=\frac{\pi}{3}}\int_0^\infty t^{s-1}\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\sin((n+1)\theta)}{\sin(\theta)n!}(-t)^n\right)dt$$

by Ramanujan's master theorem:

$$f(x)=\sum_{n=0}^{\infty}\frac{\varphi(n)}{n!}(-x)^n\Rightarrow\mathcal{M}\{f(x)\}(s)=\Gamma(s)\varphi(-s),$$

apply for Ω with $\varphi(n)=\Gamma(n+1)\sin((n+1)\theta)$

$$\Rightarrow\Omega=\frac{4}{9}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3},\theta=\frac{\pi}{3}}\Gamma(s)\Gamma(1-s)\frac{\sin((1-s)\theta)}{\sin(\theta)}=\frac{8\pi}{9\sqrt{3}}\frac{\partial}{\partial s}\bigg|_{s=\frac{2}{3}}\frac{\sin\left((1-s)\frac{\pi}{3}\right)}{\sin(\pi s)}$$

$$=\frac{8\pi}{9\sqrt{3}}\left(-\frac{\pi}{3\sin(\pi s)}\left(\cos\left(\frac{\pi}{3}(s-1)\right)+3\cos\left(\frac{\pi}{6}(2s+1)\right)\cot(\pi s)\right)\right)_{s=\frac{2}{3}}=$$

$$=-\frac{16\pi^2}{81}\left(\cos\left(\frac{\pi}{9}\right)-\sqrt{3}\sin\left(\frac{\pi}{9}\right)\right)$$

$$=-\frac{32\pi^2}{81}\left(\frac{1}{2}\cos\left(\frac{\pi}{9}\right)-\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{9}\right)\right)=-\frac{32\pi^2}{81}\left(\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{9}\right)-\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{9}\right)\right)$$

$$=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{6}-\frac{\pi}{9}\right)=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{81}\right),\text{ hence proved.}$$

Solution 4 by Ankush Kumar Parcha-India

We have,

$$\int_{\mathbb{R}^+}\frac{\ln(x)}{x^3+x\sqrt{x}+1}dx\overset{x\to x^2}{\Rightarrow}4\int_{\mathbb{R}^+}\frac{x\ln(x)}{x^6+x^3+1}dx$$

$$\Rightarrow 4\int_0^1\frac{(x-x^4)\ln(x)}{1-x^9}dx+4\underbrace{\int_1^\infty\frac{(x-x^4)\ln(x)}{1-x^9}dx}_{x\to\frac{1}{x}}$$

$$\Rightarrow 4\int_0^1\frac{(x^6-x^4-x^3+x)\ln(x)}{1-x^9}dx\overset{x^9\to x}{\Rightarrow}\frac{4}{81}\int_0^1\frac{\left(x^{-\frac{7}{9}}-x^{-\frac{5}{9}}-x^{-\frac{4}{9}}+x^{-\frac{2}{9}}\right)\ln(x)}{1-x}dx$$

$$\left(\because \psi^{(m)}(z)=-\int_0^1 \frac{t^{z-1}\ln^m(t)}{1-t}dt, \mathcal{R}(z)>0 \wedge m>0\right)$$

$$\Rightarrow \frac{4}{81}\left[\psi^{(1)}\left(\frac{5}{9}\right)+\psi^{(1)}\left(\frac{4}{9}\right)-\psi^{(1)}\left(\frac{7}{9}\right)-\psi^{(1)}\left(\frac{2}{9}\right)\right]$$

$$\left(\because \psi^{(1)}(1-z)+\psi^{(1)}(z)=\frac{\pi^2}{\sin^2(\pi z)}\right)$$

$$\Rightarrow \frac{4}{81}\left(\frac{\pi^2}{\sin^2\left(\frac{4\pi}{9}\right)}-\frac{\pi^2}{\sin^2\left(\frac{2\pi}{9}\right)}\right) \overset{\because \cos(2x)=1-2\sin^2(x)}{\Longrightarrow} \frac{4\pi^2}{81}\left(\frac{1}{\cos^2\left(\frac{\pi}{18}\right)}-\frac{2}{1-\sin\left(\frac{\pi}{18}\right)}\right)$$

$$\Rightarrow -\frac{4\pi^2}{81}\left(\frac{1+2\sin\left(\frac{\pi}{18}\right)}{\cos^2\left(\frac{\pi}{18}\right)}\right) \overset{\because \sin(3x)=3\sin(x)-4\sin^3(x)}{\Longrightarrow}$$

$$\Rightarrow \int_{\mathbb{R}^+} \frac{\ln(x)}{x^3+x\sqrt{x}+1}dx=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{18}\right)$$

Solution 5 by Djamel Arrouche-Algeria

$$\int_0^\infty \frac{\ln(x)}{x^3+x\sqrt{x}+1}dx=\Delta=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{18}\right)$$

$$x\sqrt{x}=y; dx=\frac{2}{3}y^{-\frac{1}{3}} \Rightarrow$$

$$\Delta=\frac{4}{9}\int_0^\infty \frac{\ln(y)\,y^{-\frac{1}{3}}dy}{y^2+y+1}=\frac{4}{9}\left(\int_0^1 \frac{y^{-\frac{1}{3}}\ln(y)}{y^2+y+1}+\int_1^\infty \frac{y^{-\frac{1}{3}}\ln(y)}{y^2+y+1}dy_{y\to\frac{1}{z}}\right)$$

$$=\frac{4}{9}\int_0^1 \frac{y^{-\frac{1}{3}}\ln(y)}{y^2+y+1}-\frac{4}{9}\int_0^1 \frac{z^{\frac{1}{3}}\ln(z)\,dz}{z^2+z+1}=\frac{4}{9}\int_0^1 \frac{y^{\frac{1}{3}}-y^{-\frac{1}{3}}}{y^2+y+1}\ln(y)\,dy$$

$$=\frac{4}{9}\left(f\left(-\frac{1}{3}\right)+f\left(-\frac{1}{3}\right)\right)\Bigg| f(s)=\int_0^1 \frac{y^s\ln(y)}{y^2+y+1}=\int_0^1 \frac{y^s-y^{s+1}}{1-y^3}\ln(y)$$

$$y^3=z \Rightarrow f(s)=\frac{1}{9}\int_0^1 \frac{z^{\frac{s-2}{3}}\ln(z)}{1-z}-\frac{1}{9}\int_0^1 \frac{z^{\frac{s-1}{3}}\ln(z)}{1-z}$$

we have $\Psi^{(1)}(1+z)=\int_0^1 \frac{-\ln(x)x^z}{1-x}dx$

$$\Rightarrow f(s)=\frac{1}{9}\left(\Psi^{(1)}\left(\frac{s+2}{3}\right)-\Psi^{(1)}\left(\frac{s+1}{3}\right)\right)$$

$$\Delta=\frac{4}{9}\left(f\left(-\frac{1}{3}\right)+f\left(\frac{1}{3}\right)\right)=\frac{4}{9}\cdot\frac{1}{9}\left(-\Psi^1\left(\frac{7}{9}\right)+\Psi^1\left(\frac{4}{9}\right)-1+\Psi^1\left(\frac{5}{9}\right)-\Psi^1\left(\frac{2}{9}\right)\right)$$

$$=\frac{4}{81}\left(-\left\{\Psi^1\left(\frac{14}{18}\right)+\Psi^1\left(\frac{4}{18}\right)\right\}+\left(\Psi^1\left(\frac{8}{18}\right)+\Psi^1\left(\frac{10}{18}\right)\right)\right)$$

$$\Psi^1(1-z)+\Psi^1(z)=\frac{\pi^2}{\sin^2(\pi z)}$$

$$\Delta=\frac{4\pi^2}{81}\left(-\frac{1}{\sin^2\left(\frac{4\pi}{18}\right)}+\frac{1}{\sin^2\left(\frac{8\pi}{18}\right)}\right)$$

$$\frac{1}{\sin^2\left(\frac{8\pi}{18}\right)}-\frac{1}{\sin^2\left(\frac{4\pi}{18}\right)}=-8\sin\left(\frac{\pi}{18}\right)..?$$

$$\Leftrightarrow\frac{1-4\cos^2\left(\frac{4\pi}{18}\right)}{\sin^2\left(\frac{8\pi}{18}\right)}=-8\sin\left(\frac{\pi}{18}\right)$$

$$\Leftrightarrow=\frac{1-2\left(\cos\left(\frac{8\pi}{18}\right)+1\right)}{\cos^2\left(\frac{\pi}{18}\right)}=\frac{-1-2\sin\left(\frac{\pi}{18}\right)}{\cos^2\left(\frac{\pi}{18}\right)}=-8\sin\left(\frac{\pi}{18}\right)$$

$$\Leftrightarrow -1-2\sin\left(\frac{\pi}{18}\right)=8\sin\frac{\pi}{18}\cos^2\frac{\pi}{18}=-4\sin\frac{\pi}{18}\left(\cos\left(\frac{2\pi}{18}\right)+1\right)$$

$$\Leftrightarrow -1+2\sin\left(\frac{\pi}{18}\right)=-2\left(2\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{2\pi}{18}\right)\right)=-2\left(\sin\left(\frac{\pi}{18}+\frac{2\pi}{18}\right)\right)+\sin\left(\frac{\pi}{18}-\frac{2\pi}{18}\right)$$

$$\Leftrightarrow -1+2\sin\left(\frac{\pi}{18}\right)=-2\sin\left(\frac{\pi}{6}\right)-2\sin\left(-\frac{\pi}{18}\right)\text{.. True}$$

$$\Rightarrow\frac{1}{\sin^2\left(\frac{8\pi}{18}\right)}-\frac{1}{\sin^2\left(\frac{4\pi}{18}\right)}=-8\sin\left(\frac{\pi}{18}\right)$$

$$\int_0^\infty\frac{\ln(x)}{x^2+x\sqrt{x}+1}=\frac{4\pi^2}{81}\left(\frac{1}{\sin^2\left(\frac{8\pi}{18}\right)}-\frac{1}{\sin^2\left(\frac{4\pi}{18}\right)}\right)=-\frac{32\pi^2}{81}\sin\left(\frac{\pi}{18}\right)$$