UP.528 If $a_n > 0; r_n > 0; a_{n+1} = a_n + n \cdot r_n; n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} r_n = r > 0$$

then find:

$$\Omega = \lim_{n\to\infty}(2H_n - \log a_n)$$

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Solution 1 by proposers

$$\lim_{n\to\infty}\frac{a_n}{n^2} \overset{CESARO-STOLZ}{=} \lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)^2-n^2} =$$

$$= \lim_{n\to\infty}\frac{a_{n+1}-a_n}{2n+1} = \lim_{n\to\infty}\frac{nr_n}{2n+1} = \lim_{n\to\infty}\frac{n}{2n+1}\cdot r = \frac{r}{2}$$

$$\Omega = \lim_{n\to\infty}(2H_n - \log a_n) = \lim_{n\to\infty}(2H_n - 2\log n + 2\log n - \log a_n) =$$

$$= 2\lim_{n\to\infty}(H_n - \log n) + \lim_{n\to\infty}\log\left(\frac{n^2}{a_n}\right) = 2\gamma + \log\left(\frac{2}{r}\right)$$

Solution 2 by Djamel Arrouche-Algeria

$$a_{n+1} = a_n + n\cdot r_n, \qquad \lim_{n\to\infty} r_n = r$$

$$\Omega = \lim_{n\to\infty}(2H_n - \log(a_n)), \qquad a_{n+1} - a_n = n\cdot r_n$$

$$\frac{a_{n+1}-a_n}{n} = r_n$$

$$\forall\epsilon > 0\ \exists N, \forall n \geq N\ r-\epsilon < r_n < r+\epsilon \Rightarrow n(r-\epsilon) < a_{n+1} - a_n < (r+\epsilon)n$$

$$\Rightarrow \sum_{N}^{n} k(r-\epsilon) < \sum_{k=N}^{n}(a_{k+1}-a_k) < \sum_{k=N}^{n}(r+\epsilon)\,k$$

$$\Rightarrow (r-\epsilon)\frac{(n+N)(n-N+1)}{2} < a_{n+1} - a_N < (r+\epsilon)\frac{(n-N+1)(n+N)}{2}$$

$$\Rightarrow \left(1-\frac{\epsilon}{r}\right)\left(1+\frac{n}{N}\right)\left(1+\frac{1-N}{n}\right) < \frac{2}{n^2 r}(a_{n+1}-a_N) <$$

$$< \left(1+\frac{\epsilon}{r}\right)\left(1+\frac{n}{N}\right)\left(1+\frac{1-N}{n}\right)$$

ϵ is arbitrary small

$$\Rightarrow \lim_{n\to\infty}\frac{2}{n^2 r}(a_{n+1}-a_N) = 1$$

Since $a_N \in \mathbb{R} \Rightarrow \lim_{n\to\infty} a_{n+1}\cdot\frac{2}{n^2 r} = 1$

$$2H_n - \log(a_n) = 2H_n + \log\left(\frac{2}{n^2 r}\right) - \log\left(\frac{2}{n^2 r} a_n\right) =$$

$$= 2H_n - 2\log(n) + \log(2) + \log\left(\frac{2}{n^2 r} a_n\right)$$

$$= 2(H_n - \log(n)) + \log\left(\frac{2}{r}\right) + \log\left(\frac{2}{n^2 r} a_n\right)$$

$\lim_{n\to\infty} \log\left(\frac{a_n \cdot 2}{n^2 r}\right) = \log(1) = 0; \lim_{n\to\infty} H_n - \log(n) = \gamma$: Euler Mascheroni Constant

$$\Omega = \lim_{n\to\infty}(2H_n - \log(a_n)), \qquad \Omega = 2\cdot\gamma + \log\left(\frac{2}{r}\right)$$

Solution 3 by Kamel Gandouli Rezgui-Algeria

$$a_{n+1} - a_n = nr_n; n > 0$$

$$\Rightarrow \sum_{k=1}^{n-1} a_{k+1} - a_k = \sum_{k=1}^{n-1} kr_k \Rightarrow a_n = \sum_{k=1}^{n-1} kr_k + a_1$$

$$\Rightarrow \log(a_n) = \log\left(\sum_{k=1}^{n-1} kr_k + a_1\right) = \log n^2 + \log\left(\frac{1}{n^2}\sum_{k=1}^{n-1} kr_k + \frac{a_1}{n^2}\right)$$

$$= \log n^2 + \log\left(\frac{1}{n^2}\sum_{k=1}^{n} kr_k - \frac{r_n}{n} + \frac{a_1}{n^2}\right)$$

$$\frac{1}{n^2}\sum_{k=1}^{n} k(r_k - r) + \frac{1}{n^2}\sum_{k=1}^{n} kr$$

$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} k(r_k - r) = 0$ Cesaro lemma$\Rightarrow \lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} k(r_k - r) = 0$

and $\frac{1}{n^2}\sum_{k=1}^{n} kr = \frac{n(n+1)}{2n^2} r \Rightarrow \lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} kr = \frac{r}{2}$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} kr_k = \frac{r}{2} \Rightarrow \lim_{n\to+\infty} \log\left(\frac{1}{n^2}\sum_{k=1}^{n} kr_k - \frac{r_n}{n} + \frac{a_1}{n^2}\right) = \log\frac{r}{2}$$

$$\Rightarrow 2H_n - \log a_n = 2H_n - 2\log n + \log\left(\frac{1}{n^2}\sum_{k=1}^{n} kr_k - \frac{r_n}{n} + \frac{a_1}{n^2}\right)$$

$$\Rightarrow \lim_{n\to+\infty} 2H_n - \log a_n = 2\gamma + \log\frac{r}{2}$$

Solution 4 by Le Thu-Vietnam

$$\because a_{n+1} = a_n + nr_n \Rightarrow a_{n+2} - a_{n+1} = (n+1)r_{n+1}$$

$$\because \lim r_{n+1} = \lim r_n = r \in \mathbb{R} \Rightarrow \lim \frac{r_n}{n} = 0$$

$$\therefore \Omega \equiv \lim[2\mathcal{H}_n - \ln(a_n)]$$

$$= \lim[2\mathcal{H}_n - \ln(a_{n+1} - nr_n)]$$

$$= 2\overbrace{\lim[\mathcal{H}_n - \ln(n)]}^{\equiv\gamma} - \lim\ln\left\{\frac{a_{n+1}}{n^2} - \frac{r_n}{n}\right\} = 2\gamma - \ln\left(\lim\frac{a_{n+1}}{n^2}\right)$$

$$\stackrel{s-c}{=} 2\gamma - \ln\left\{\lim\frac{a_{n+2} - a_{n+1}}{(n+1)^2 - n^2}\right\} = 2\gamma - \ln\left(r \cdot \lim\frac{n+1}{2n+1}\right) = 2\gamma - \ln\left(\frac{r}{2}\right)$$

Solution 5 by Pham Duc Nam-Vietnam

If: $a_n > 0; r_n > 0, a_{n+1} = a_n + nr_n (\forall n \in \mathbb{N}^*)$ and $\lim\limits_{n\to\infty} r_n = r > 0$

Then find: $\Omega = \lim\limits_{n\to\infty}(2H_n - \ln(a_n))$

$$\lim_{n\to\infty} r_n = r \Rightarrow \lim_{n\to\infty} r_{n+1} = r;\ \lim_{n\to\infty}\frac{\sum_{k=1}^{n} kr_k}{n^2} = \frac{r}{2}$$

Indeed, by Stolz – Cesaro theorem: $\lim\limits_{n\to\infty}\frac{\sum_{k=1}^{n+1} kr_k - \sum_{k=1}^{n} kr_k}{(n+1)^2 - n^2} = \lim\limits_{n\to\infty}\frac{(n+1)r_{n+1}}{2n+1} = \frac{r}{2}$

For $k \le n, a_{k+1} - a_k = kr_k \Rightarrow \underbrace{\sum_{k=1}^{n}(a_{k+1} - a_k)}_{Telescoping\ sum} = \sum_{k=1}^{n} kr_k \Leftrightarrow a_{n+1} - a_1 = \sum_{k=1}^{n} kr_k$

$$\Leftrightarrow a_{n+1} - a_n + a_n - a_1 = \sum_{k=1}^{n} kr_k \Leftrightarrow a_n = \sum_{k=1}^{n} kr_k + a_1 - nr_n \Rightarrow$$

$$\Rightarrow \frac{a_n}{n^2} = \frac{\sum_{k=1}^{n} kr_k}{n^2} + \frac{a_1}{n^2} - \frac{r_n}{n} \Rightarrow \ln(a_n) = \ln\left(\frac{\sum_{k=1}^{n} kr_k}{n^2} + \frac{a_1}{n^2} - \frac{r_n}{n}\right) + 2\ln(n)$$

$$\Omega = \lim_{n\to\infty}(2H_n - \ln(a_n)) = \lim_{n\to\infty}\left(2H_n - 2\ln(n) - \ln\left(\frac{\sum_{k=1}^{n} kr_k}{n^2} + \frac{a_1}{n^2} - \frac{r_n}{n}\right)\right) =$$

$$= \lim_{n\to\infty}\left(2H_n - 2\ln(n) - \ln\left(\frac{r}{2}\right)\right)$$

By definition: $\gamma = \lim\limits_{n\to\infty}(H_n - \ln(n)) \Rightarrow \Omega = 2\gamma - \ln\left(\frac{r}{2}\right)$