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UP.529 If $a_n > 0; n \in \mathbb{N}^*; \lim_{n\to\infty}(a_{n+1} - a_n) = a > 0$ then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot a_n^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

We will use Lalescu's sequence:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{1}{e}$$

$$\lim_{n\to\infty}\frac{a_n}{n} \overset{CESARO-STOLZ}{=} \lim_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n} = \lim_{n\to\infty}(a_{n+1}-a_n) = a$$

$$\Omega = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\right)\cdot a_n^2 =$$

$$= \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\left(\frac{a_n}{n}\right)^2\cdot\frac{n}{n+1} =$$

$$= \frac{1}{e}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\lim_{n\to\infty}\left(\frac{a_n}{n}\right)^2\cdot\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{e}\cdot e\cdot e\cdot a^2\cdot 1 = a^2 e$$

Solution 2 by Angel Plaza-Spain

$\lim_{n\to\infty}\frac{a_{n+1}}{n+1} = \lim_{n\to\infty}(a_{n+1}-a_n) = a$. **Therefore** $\lim_{n\to\infty}\frac{a_n^2}{n^2} = \lim_{n\to\infty}\left(\frac{a_n}{n}\right)^2 = a^2$. **Hence**

$$\Omega = a^2\cdot\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot n^2$$

By using that [1]:

$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) = e$, **then, by Stirling formula for n!**

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot n^2 = -e + \lim_{n\to\infty}\frac{2n+1}{\sqrt[n+1]{(n+1)!}}$$

$$= -e + \lim_{n\to\infty}\frac{2n+1}{(n+1)e^{-1}} = -e + 2e = e.$$

Therefore: $\Omega = a^2 e$

Reference:

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[1] D.M. Bătinețu – Giurgiu, Daniel Sitaru,Neculai Stanciu- 120 Years of Lalescu Sequences, RMM 29 (2021), Available at https://www.ssmrmh.ro/wp-content/uploads/2020/02/120-YEARS-OF-LALESCU-SEQUENCES.pdf