ROMANIAN MATHEMATICAL MAGAZINE

UP.530 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}} - \frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot e^{2H_n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

Solution 1 by proposers

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n} =$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}\cdot\frac{(2n-1)!!}{(2n-1)!!}\cdot\frac{n+1}{2n+1} = \frac{e}{2}$$

$$\lim_{n\to\infty}\frac{e^{H_n}}{n} = \lim_{n\to\infty} e^{-\log n + H_n} = e^{\gamma}$$

$$\lim_{n\to\infty}\frac{e^{2H_n}}{n^2} = e^{2\gamma}$$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!}\right)\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{e^{2H_n}}{n^2}\cdot\frac{n+1}{n} =$$

$$= \frac{2}{e}\cdot\frac{e}{2}\cdot\frac{e}{2}\cdot e^{2\gamma}\cdot 1 = \frac{1}{2}e^{2\gamma+1}$$

Solution 2 by Angel Plaza-Spain

$\lim_{n\to\infty}\frac{e^{H_n}}{n} = \lim_{n\to\infty}\frac{e^{H_n}}{e^{\ln n}} = \lim_{n\to\infty} e^{H_n-\ln n} = e^{\gamma}$. Therefore, $\lim_{n\to\infty}\frac{e^{2H_n}}{n^2} = \lim_{n\to\infty}\left(\frac{e^{H_n}}{n}\right)^2 = e^{2\gamma}$

Hence, $\Omega = e^{2\gamma}\cdot\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}} - \frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot n^2$

By using that [1]

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!}} - \frac{n^2}{\sqrt[n]{(2n+1)!!}}\right) = \frac{e}{2}$$

Then,

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}} - \frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot n^2 = -\frac{e}{2} + \lim_{n\to\infty}\frac{2n+1}{\sqrt[n+1]{(2n+1)!!}}$$

$$= -\frac{e}{2} + \lim_{n\to\infty}\frac{(2n+1)\sqrt[n+1]{(2n+2)!!}}{\sqrt[n+1]{(2n+2)!}}$$

$$= -\frac{e}{2} + \lim_{n\to\infty} \frac{(2n+1)\sqrt[n+1]{2^{n+1}(n+1)^{n+1}e^{-n-1}}}{\sqrt[n+1]{(2n+2)^{2n+2}e^{-2n-2}}}$$

$$= -\frac{e}{2} + \lim_{n\to\infty} \frac{(2n+1)2(n+1)e^{-1}}{(2n+2)^2e^{-2}} = -\frac{e}{2} + e = \frac{e}{2}$$

Therefore:

$$\Omega = e^{2\gamma} \cdot \frac{e}{2} = \frac{1}{2}e^{1+2\gamma}$$

Reference:

[1] D.M. Bătinețu - Giurgiu, Daniel Sitaru,Neculai Stanciu- 120 Years of Lalescu Sequences, RMM 29 (2021). Available at https://www.ssmrmh.ro/wp-content/uploads/2020/02/120-YEARS-OF-LALESCU-SEQUENCES.pdf