ROMANIAN MATHEMATICAL MAGAZINE

UP.531 Prove that:

$$\int_0^{\infty}\frac{x^2\sinh(2x)}{\cosh^2(2x)}dx=\frac{3\pi^3}{128}$$

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Solution by proposer

The following problems are crucial to our solution

Problem 1. Prove

$$\beta(3)=\frac{\pi^3}{32},$$

with $\beta(s)$ design the Dirichlet beta function defined as

$$\beta(s)=\sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\mathcal{R}(s)>0.$$

Problem 2. Prove

$$\int_0^{\infty}\frac{x^2}{\cosh(x)}dx=\frac{\pi^3}{8}.$$

Proof of Problem 1. We have

$$\beta(3)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{(2(2n)+1)^3}+\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{(2(2n+1)+1)^3}=\frac{1}{64}\left(\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{4}\right)^3}-\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{3}{4}\right)^3}\right)$$

$$=-\frac{1}{128}\left(\psi^{(2)}\left(\frac{1}{4}\right)-\psi^{(2)}\left(\frac{3}{4}\right)\right),$$

where $\psi(x)$ is the digamma function and $\psi^{(k)}(x)$ is its derivatives of order $k\geq 1$ given as:

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+x}\right)\ x>0;\ \psi^{(k)}(x)=(-1)^{k+1}k!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{k+1}}.$$

Applying a reflection formula for the general $k-$th order polygamma function $\psi^{(k)}(x)$, that is

$$\psi^{(k)}(1-x)+(-1)^{k+1}\psi^{(k)}(x)=(-1)^k\pi\frac{\partial^k}{\partial x^k}\cot(\pi x),$$

we can deduce for $x=\frac{3}{4}$:

$$\psi^{(2)}\left(\frac{1}{4}\right)-\psi^{(2)}\left(\frac{3}{4}\right)=\pi\left[\frac{\partial^2}{\partial x^2}\left(\frac{\cos(\pi x)}{\sin(\pi x)}\right)\right]_{x=\frac{3}{4}}=\pi\left[\frac{\partial}{\partial x}\left(\frac{-\pi}{\sin^2(\pi x)}\right)\right]_{x=\frac{3}{4}}$$

$$=\pi\left[\frac{2\pi^2\cot(\pi x)}{\sin^2(\pi x)}\right]_{x=\frac{3}{4}}=\frac{2\pi^3(-1)}{\left(\frac{1}{\sqrt{2}}\right)^2}=-4\pi^3$$

Thereby

$$\beta(3)=-\frac{1}{128}(-4\pi^3)=\frac{\pi^3}{32}.$$

Remark. It's worth noting that the result of Problem 1 exist without proof in mathworld.wolfram.com (1) and wikipedia.org (2)

(1) https://mathworld.wolfram.com/DirichletBetaFunction.html

(2) https://en.wikipedia.org/wiki/Dirichlet_beta_function

Proof of Problem 2.

Let $J=\int_0^\infty \frac{x^2}{\cosh(x)}dx$, so we have

$$J=2\int_0^\infty \frac{x^2}{e^x+e^{-x}}dx=2\int_0^\infty \frac{x^2e^{-x}}{1+e^{-2x}}dx=2\int_0^\infty x^2e^{-x}\left(\sum_{n=0}^\infty(-1)^n e^{-2nx}\right)dx$$

$$=2\left(\sum_{n=0}^\infty(-1)^n\int_0^\infty x^2e^{-(2n+1)x}\,dx\right)=2\left(\sum_{n=0}^\infty\frac{(-1)^n}{(2k+1)^3}\int_0^\infty t^2e^{-t}\,dt\right)$$

$$=2\left(\sum_{n=0}^\infty\frac{(-1)^n\Gamma(3)}{(2n+1)^3}\right)=4\left(\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^3}\right)=4\beta(3)=4\left(\frac{\pi^3}{32}\right)=\frac{\pi^3}{8}$$

Now, replacing x by $ax, a>0$ in the integral J, we obtain

$$\frac{\pi^3}{8}=\int_0^\infty\frac{(ax)^2}{\cosh(ax)}a\,dx,$$

which imply

$$\frac{\pi^3}{8a^3}=\int_0^\infty\frac{x^2}{\cosh(ax)}dx.$$

Differentiate both sides with respect to a, we get

$$-\frac{3\pi^3}{8a^3} = \int_0^\infty x^2 \frac{\partial}{\partial a}\left(\frac{1}{\cosh(ax)}\right) dx = \int_0^\infty x^2 \left(-\frac{x\sinh(ax)}{\cosh^2(ax)}\right) dx$$

Hence

$$\frac{3\pi^3}{8a^4} = \int_0^\infty \frac{x^2\sinh(ax)}{\cosh^2(ax)} dx$$

Fix $a = 2$, we finally have

$$\int_0^\infty \frac{x^2\sinh(2x)}{\cosh^2(2x)} dx = \frac{3\pi^3}{128}.$$