ROMANIAN MATHEMATICAL MAGAZINE

UP.532 Find all continuous functions $f:\mathbb{R}\to\mathbb{R}; f(0)=0$ such that:

$$f(x)=f\left(\frac{x}{5}\right)+\frac{x}{7};(\forall)x\in\mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Replacing x successively with: $\frac{x}{5};\frac{x}{5^2};\dots;\frac{x}{5^{n-1}}$

$$f(x)-f\left(\frac{x}{5}\right)=\frac{x}{7}$$

$$f\left(\frac{x}{5}\right)-f\left(\frac{x}{5^2}\right)=\frac{x}{7\cdot 5}$$

$$f\left(\frac{x}{5^2}\right)-f\left(\frac{x}{5^3}\right)=\frac{x}{7\cdot 5^2}$$

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$$f\left(\frac{x}{5^{n-1}}\right)-f\left(\frac{x}{5^n}\right)=\frac{x}{7\cdot 5^{n-1}}$$

By adding:

$$f(x)-f\left(\frac{x}{5^n}\right)=\frac{x}{7}\left(1+\frac{1}{5}+\frac{1}{5^2}+\cdots+\frac{1}{5^{n-1}}\right)$$

$$f(x)-f\left(\frac{x}{5^n}\right)=\frac{x}{7}\cdot\frac{\frac{1}{5^n}-1}{\frac{1}{5}-1}$$

$$\lim_{n\to\infty}\left(f(x)-f\left(\frac{x}{5^n}\right)\right)=\lim_{n\to\infty}\frac{x}{7}\cdot\frac{\frac{1}{5^n}-1}{\frac{1}{5}-1}$$

$$f(x)-f(0)=\frac{x}{7}\cdot\frac{0-1}{-\frac{4}{5}},\qquad f(x)-0=\frac{5x}{28}\Rightarrow f(x)=\frac{5x}{28}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $n\in\mathbb{N}$. For all $k\in\{0,1,\dots,n-1\}$, if we replace x by $\frac{x}{5^k}$, we obtain

$$f\left(\frac{x}{5^k}\right)-f\left(\frac{x}{5^{k+1}}\right)=\frac{x}{7.5^k},\text{ then }\sum_{k=0}^{n-1}\left(f\left(\frac{x}{5^k}\right)-f\left(\frac{x}{5^{k+1}}\right)\right)=\sum_{k=0}^{n-1}\frac{x}{7.5^k}$$

$$\text{or } f(x) = f\left(\frac{x}{5^n}\right) + \frac{x}{7}\cdot\frac{1-\left(\frac{1}{5}\right)^n}{\frac{4}{5}}, \forall x \in \mathbb{R}.$$

Since f is continuous on $\mathbb{R}$, then $\lim_{n\to\infty} f\left(\frac{x}{5^n}\right) = f\left(\lim_{n\to\infty}\frac{x}{5^n}\right) = f(0) = 0$.

If we tend n to infinity in the last relation and since $\lim_{n\to\infty}\left(\frac{1}{5}\right)^n = 0$, we obtain

$$f(x) = \frac{5x}{28}, \quad \forall x \in \mathbb{R},$$

The function satisfies the conditions of the problem.