ROMANIAN MATHEMATICAL MAGAZINE

UP.533 Calculate the integral:

$$\int_{-1}^{1} \frac{\arccos x}{\sqrt{4x^4 + x^2 + 4}} dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

We make the notation: $A = \int_{-1}^{1} \frac{\arccos x}{\sqrt{4x^4+x^2+4}} dx$.

We also consider the integral: $B = \int_{-1}^{1} \frac{\arcsin x}{\sqrt{4x^4+x^2+4}} dx$

We have:

$$A + B = \int_{-1}^{1} \frac{\arccos x + \arcsin x}{\sqrt{4x^4 + x^2 + 4}} dx = \frac{\pi}{2} \int_{-1}^{1} \frac{1}{\sqrt{4x^4 + x^2 + 4}} dx$$

But, we have:

$$\int_{-1}^{1} \frac{1}{\sqrt{4x^4 + x^2 + 4}} dx = 2 \int_{0}^{1} \frac{1}{\sqrt{4x^4 + x^2 + 4}} dx$$

because the function under the integral sign is even.

We are going to calculate the integral:

$$C = \int_{0}^{1} \frac{1}{\sqrt{4x^4 + x^2 + 4}} dx$$

We will show that the C integral can be expressed using the complete elliptic integral of the first kind.

The complete elliptic integral of the first kind is defined by the relationship:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2 \sin^2 \theta}} d\theta, \text{ with } -1 < k < 1.$$

Substitute:

$$t = \tan \frac{\theta}{2}, \text{ so } \sin \theta = \frac{2t}{1+t^2} \text{ and } d\theta = \frac{2}{1+t^2} dt.$$

We have:

$$K(k)=\int_0^1 \frac{1}{\sqrt{1-k^2\frac{4t^2}{(1+t^2)^2}}}2\frac{1}{1+t^2}dt=2\int_0^1 \frac{1}{\sqrt{t^4+(2-4k^2)t^2+1}}dt$$

We have:

$$C=\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x^4+\frac{1}{4}x^2+1}}dx$$

We put the condition:

$2-4k^2=\frac{1}{4}$, so $k=\frac{\sqrt{7}}{4}$ (we choose $k>0$).

We obtain:

$$C=\frac{1}{2}\cdot\frac{1}{2}\cdot K\left(\frac{\sqrt{7}}{4}\right)$$

The integral B is equal to zero, because the function under the integral sign is odd.

So, we have: $A=\frac{\pi}{2}\cdot 2C$

We obtained the value of the integral required in the problem statement:

$$A=\frac{\pi}{4}K\left(\frac{\sqrt{7}}{4}\right).$$