ROMANIAN MATHEMATICAL MAGAZINE

UP.534 If $a, b > 0$ then:

$$\int_a^b \int_a^b \left(\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\right) dxdy \le \ln^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer, Solution 2 by Tapas Das – India

Solution 1 by proposer

$$\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2} \overset{AM-GM}{\le} \frac{x}{2\sqrt{x^4y^2}}+\frac{y}{2\sqrt{y^4x^2}}=$$

$$=\frac{x}{2x^2y}+\frac{y}{2y^2x}=\frac{1}{2xy}+\frac{1}{2xy}=\frac{1}{xy}$$

$$\int_a^b \int_a^b \left(\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\right) dxdy \le \int_a^b \int_a^b \frac{1}{xy} dxdy =$$

$$=\left(\int_a^b \frac{1}{x}dx\right)\left(\int_a^b \frac{1}{y}dy\right)=(\ln b-\ln a)^2=\ln^2\left(\frac{b}{a}\right)$$

Equality holds for $a = b$.

Solution 2 by Tapas Das – India

$$\frac{x}{x^4+y^2} \overset{AM-GM}{\le} \frac{x}{2\sqrt{x^4y^2}}=\frac{1}{2xy}$$

$$\frac{y}{y^4+x^2} \le \frac{y}{2\sqrt{y^4x^2}}=\frac{1}{2xy}$$

$$\therefore \frac{x}{x^4+y^2}+\frac{y}{y^4+x^2} \le \frac{1}{2xy}+\frac{1}{2xy}=\frac{1}{xy}$$

$$\int_a^b \int_a^b \left[\left(\frac{x}{x^4+y^2}\right)+\left(\frac{y}{y^4+x^2}\right)\right] dx\, dy \le \int_a^b \int_a^b \frac{1}{xy} dx\, dy$$

$$=\int_a^b \frac{1}{x}dx \cdot \int_a^b \frac{1}{y}dy=(\ln b-\ln a)^2=\ln^2\frac{b}{a}$$