ROMANIAN MATHEMATICAL MAGAZINE

UP.535 If $x, y, z > 1; x \neq y \neq z \neq x$ and

$$\log_{\frac{y}{z}} x + \log_{\frac{z}{x}} y + \log_{\frac{x}{y}} z = 0$$

then:

$$\frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)} + \frac{\log_2 y}{\log_2^2\left(\frac{z}{x}\right)} + \frac{\log_2 z}{\log_2^2\left(\frac{x}{y}\right)} = 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$0 = \sum_{cyc} \log_{\frac{y}{z}} x = \left(\sum_{cyc} \log_{\frac{y}{z}} x\right) \cdot \left(\sum_{cyc} \frac{1}{\log_2 \frac{y}{z}}\right) =$$

$$= \left(\sum_{cyc} \frac{\log_2 x}{\log_2 y - \log_2 z}\right) \cdot \left(\sum_{cyc} \frac{1}{\log_2 y - \log_2 z}\right) =$$

$$= \sum_{cyc} \frac{\log_2 x}{(\log_2 y - \log_2 z)^2} + \sum_{cyc} \frac{\log_2 y + \log_2 z}{(\log_2 z - \log_2 x)(\log_2 x - \log_2 y)} =$$

$$= \sum_{cyc} \frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)} + \sum_{cyc} \frac{(\log_2 y + \log_2 z)(\log_2 y - \log_2 z)}{(\log_2 z - \log_2 x)(\log_2 x - \log_2 y)(\log_2 y - \log_2 z)} =$$

$$= \sum_{cyc} \frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)} + \frac{1}{\prod_{cyc}(\log_2 z - \log_2 x)} \cdot \sum_{cyc} (\log_2^2 y - \log_2^2 z) =$$

$$= \sum_{cyc} \frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)} + \frac{1}{\prod_{cyc}(\log_2 z - \log_2 x)} \cdot 0 = \sum_{cyc} \frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a = \ln x, b = \ln y, c = \ln z$.

The given condition is equivalent to $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0$,

and the problem becomes to prove that $\left(\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2}\right) . \ln 2 = 0$.

We have

$$\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2} = \left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)\left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b}\right) -$$

$$-\frac{a}{b-c}\left(\frac{1}{c-a} + \frac{1}{a-b}\right) - \frac{b}{c-a}\left(\frac{1}{a-b} + \frac{1}{b-c}\right) - \frac{c}{a-b}\left(\frac{1}{b-c} + \frac{1}{c-a}\right)$$

$$= 0.\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)-\frac{a(c-b)+b(a-c)+c(b-a)}{(a-b)(b-c)(c-a)} = 0,$$

which completes the proof.