UP.536 If $1 < a \leq b; m \geq 1$ then:

$$\frac{(b-1)^{m+1}-(a-1)^{m+1}}{m+1}+b-a \leq \frac{b^{m+1}-a^{m+1}}{m+1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{(b-1)^{m+1}-(a-1)^{m+1}}{m+1}=\int_a^b (x-1)^m\,dx$$

$$b-a=\int_a^b dx$$

$$\frac{b^{m+1}-a^{m+1}}{m+1}=\int_a^b x^m\,dx$$

We must prove that

$$\int_a^b (x-1)^m\,dx+\int_a^b dx \leq \int_a^b x^m\,dx$$

$$\int_a^b ((x-1)^m+1)\,dx \leq \int_a^b x^m\,dx$$

It is enough to prove that:

$$(x-1)^m+1 \leq x^m; (\forall)x>1; m \geq 1$$

Let be: $f:(1,\infty)\to\mathbb{R}; f(x)=(x-1)^m-x^m+1$

$$f'(x)=m((x-1)^{m-1}-x^{m-1})<0 \text{ because } x-1<x$$

f decreasing on $(1,\infty)$

$$\sup_{x>1} f(x)=\lim_{\substack{x\to 1\\ x>1}} f(x)=\lim_{\substack{x\to 1\\ x>1}}((x-1)^m-x^m+1)=0$$

$$\Rightarrow f(x)\leq 0; (\forall)x>1 \Rightarrow (x-1)^m-x^m+1 \leq 0$$

Equality holds for $a=b$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Write the given inequality as $f(a)\leq f(b)$, where

$$f(x)=\frac{x^{m+1}-(x-1)^{m+1}}{m+1}-x,\quad x\geq 1.$$

We have $f'(x)=x^m-(x-1)^m-1$ and

$f''(x) = m[x^{m-1} - (x-1)^{m-1}] \geq 0$, then f' is
increasing on $[1, \infty)$ and
$f'(x) \geq f'(1) = 0, \forall x \geq 1$, then f is increasing on $[1, \infty)$, and since
$1 < a \leq b$, then $f(a) \leq f(b)$, as desired.
Equality holds if $a = b$.