ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n\cdot 8^n}\binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega = \lim_{n\to\infty} \frac{1}{n\cdot 8^n}\binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2} = \lim_{n\to\infty}\frac{1}{n\cdot 8^n}\cdot\frac{(4n)!\cdot(4n)!}{n!\,(3n)!\,(2n)!\,(2n)!}\cdot\frac{(n!)^2\big((2n)!\big)^2}{\big((3n)!\big)^2} =$$

$$= \lim_{n\to\infty}\frac{1}{n\cdot 8^n}\cdot\frac{\big((4n)!\big)^2\cdot n!}{\big((3n)!\big)^3} = (*);\ but\ n! \cong \frac{n^n\sqrt{2n\pi}}{e^n}, then$$

$$(*) = \lim_{n\to\infty}\frac{[(4n)^{4n}]^2\cdot 2\cdot 4n\pi}{(e^{4n})^2}\cdot\frac{n^n\sqrt{2n\pi}}{e^n}\cdot\frac{(e^{3n})^3}{[(3n)^{3n}]^3\cdot 6n\pi\sqrt{6n\pi}}\cdot\frac{1}{n\cdot 8^n} =$$

$$= \lim_{n\to\infty}\frac{4^{2n}}{3^{9n}\cdot 3\sqrt{3n}\cdot 2^{3n}} = \lim_{n\to\infty}\frac{\left(\frac{4}{3}\right)^{8n}}{24^n\cdot 3\sqrt{3n}} = \lim_{n\to\infty}\left(\frac{\frac{4^8}{3^8}}{24}\right)^n\cdot\frac{1}{3\sqrt{3n}} = 0.$$

Solution 2 by Ravi Prakash-New Delhi-India

$$Let\ a_n = \binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2} = \frac{(4n)!\cdot(4n)!}{n!\,(3n)!\,(2n)!\,(2n)!}\cdot\frac{(n!)^2\big((2n)!\big)^2}{\big((3n)!\big)^2} =$$

$$= \frac{\big((4n)!\big)^2\cdot n!}{\big((3n)!\big)^3} = \frac{\big((3n+1)(3n+2)\dots(4n)\big)^2}{(2n+1)(2n+2)\dots(2n)} \le \frac{(4n)^{2n}}{(2n)^{2n}} = \frac{2^{4n}}{2^{2n}} = 2^{2n}$$

$$Now, 0 < \frac{1}{8^n\cdot n}a_n < \frac{1}{2^n\cdot n}. As \lim_{n\to\infty}\frac{1}{2^n\cdot n} = 0, then:$$

$$\Omega = \lim_{n\to\infty}\frac{1}{n\cdot 8^n}\binom{4n}{n}\binom{4n}{2n}\binom{3n}{n}^{-2} = 0.$$