ROMANIAN MATHEMATICAL MAGAZINE

UP.538 Solve for real numbers:

$$\int_4^x \frac{t^2+1}{t^3+1}dt = 2(\sqrt{x}-2)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Aki Le-Vietnam

$$Let\ f(x) = \int_4^x \frac{t^2+1}{t^3+1}dt - 2(\sqrt{x}-2), then$$

$$f'(x) = \frac{-(\sqrt{x}-1)(\sqrt{x^5}-1)}{(x^3+1)\sqrt{x}} \le 0, \forall x > 0$$

$$Moreover, f(4) = 0. Then\ the\ solution\ set\ is\ \{4\}.$$

Solution 2 by Christos Tsifakis-Greece

$$Let\ f(x) = \int_4^x \frac{t^2+1}{t^3+1}dt - 2(\sqrt{x}-2), x \ge 0.$$

$$For\ x = 4, we\ have\ f(4) = 0.$$

$$For\ x > 0, we\ have: f'(x) = \frac{x^2+1}{x^3+1} - \frac{1}{\sqrt{x}} = \frac{(x^2\sqrt{x}-1)(1-\sqrt{x})}{\sqrt{x}(x^3+1)} =$$

$$= \frac{-(\sqrt{x}-1)^2(\sqrt{x^4}+\sqrt{x^3}+\sqrt{x^2}+1)}{\sqrt{x}(x^3+1)} \le 0.$$

$$So, f \searrow [0,\infty)\ and\ x = 4\ is\ the\ only\ solution.$$

Solution 3 by Marin Chirciu – Romania

The condition of existence of the radical is $x \ge 0$

We decompose in simple fractions $\frac{t^2+1}{t^3+1} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$

We obtain $A = \frac{2}{3}, B = \frac{1}{3}, C = \frac{1}{3}$

$$\int \frac{t^2+1}{t^3+1}dt = \frac{2}{3}\ln(t+1) + \frac{1}{3}\int \frac{t+1}{t^2-t+1}dt$$

$$\int \frac{t+1}{t^2-t+1}dt = \frac{1}{2}\int \frac{2t-1+3}{t^2-t+1}dt = \frac{1}{2}\int \frac{2t-1}{t^2-t+1}dt + \frac{3}{2}\int \frac{1}{t^2-t+1}dt =$$

$$=\frac{1}{2}\ln(t^2-t+1)+\frac{3}{2}\int\frac{1}{t^2-t+\frac{1}{4}+\frac{3}{4}}dt=\frac{1}{2}\ln(t^2-t+1)+\frac{3}{2}\int\frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt=$$

$$=\frac{1}{2}\ln(t^2-t+1)+\frac{3}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{2}(t^2-t+1)+\sqrt{3}\arctan\frac{2t-1}{\sqrt{3}}.$$

It follows:

$$\int\frac{t^2+1}{t^3+1}dt=\frac{2}{3}\ln(t+1)+\frac{1}{3}\int\frac{t+1}{t^2-t+1}dt=$$

$$=\frac{2}{3}\ln(t+1)+\frac{1}{3}\left(\frac{1}{2}\ln(t^2-t+1)+\sqrt{3}\arctan\frac{2t-1}{\sqrt{3}}\right)=$$

$$=\frac{2}{3}\ln(t+1)+\frac{1}{6}\ln(t^2-t+1)+\frac{\sqrt{3}}{3}\arctan\frac{2t-1}{\sqrt{3}}$$

We obtain:

$$\int_4^x\frac{t^2+1}{t^3+1}dt=\left[\frac{2}{3}\ln(t+1)+\frac{1}{6}\ln(t^2-t+1)+\frac{\sqrt{3}}{3}\arctan\frac{2t-1}{\sqrt{3}}\right]_4^x=$$

$$=\frac{2}{3}\ln\frac{x+1}{5}+\frac{1}{6}\ln\frac{x^2-x+1}{13}+\frac{\sqrt{3}}{3}\arctan\frac{\frac{2x-1}{\sqrt{3}}-\frac{7}{\sqrt{3}}}{1+\frac{2x-1}{\sqrt{3}}\cdot\frac{7}{\sqrt{3}}}=$$

$$=\frac{2}{3}\ln\frac{x+1}{5}+\frac{1}{6}\ln\frac{x^2-x+1}{13}+\frac{\sqrt{3}}{3}\arctan\frac{\frac{2x-8}{\sqrt{3}}}{1+\frac{7(2x-1)}{3}}.$$

The equation $\int_4^x\frac{t^2+1}{t^3+1}dt=2(\sqrt{x}-2)$ can be written:

$\frac{2}{3}\ln\frac{x+1}{5}+\frac{1}{6}\ln\frac{x^2-x+1}{13}+\frac{\sqrt{3}}{3}\arctan\frac{\frac{2x-8}{\sqrt{3}}}{1+\frac{7(2x-1)}{3}}=2(\sqrt{x}-2)$**, with $x=4$ unique solution.**

We deduce that $x=4$ is the unique solution of the equation.

Solution 4 by Angel Plaza-Spain

By simple inspection it follows that $x=4$ is a real solution to the given equation. Let us prove that there is no solution. Consider function $f(x)$ defined by

$$f(x)=\int_4^x\frac{t^2+1}{t^3+1}dt-2(\sqrt{x}-2),$$

for $x \geq 0$. f is a continuous function for $x \geq 0$. Then by the Fundamental Theorem of Integral Calculus, $f'(x) = \frac{x^2+1}{x^3+1} - \frac{1}{\sqrt{x}} = -\frac{(\sqrt{x}-1)^2(x^2+x\sqrt{x}+x+\sqrt{x}+1)}{\sqrt{x}(x+1)(x^2-x+1)} \leq 0$ for all $x \geq 0$.

Therefore, function $f(x)$ is monotonically decreasing and it has no more than a single root in $\mathbb{R}^+$.