ROMANIAN MATHEMATICAL MAGAZINE

UP.539 If $0 < a \leq b$ then:

$$\int_a^b e^{x^2}\,dx \geq (b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban – Romania

$$e^x \geq x+1, (\forall)x > 0 \Rightarrow e^{x^2} \geq x^2+1$$

$$\Rightarrow \int_a^b e^{x^2}\,dx \geq \int_a^b (x^2+1)\,dx = \frac{b^3-a^3}{3}+b-a =$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3}+b-a = (b-a)\cdot\left(\frac{a^2+ab+b^2}{3}+1\right) \geq$$

$$\geq (b-a)\cdot\sqrt[3]{a^2+ab+b^2}$$

$$b-a \geq 0, \frac{a^2+ab+b^2}{3}+1 \geq \sqrt[3]{a^2+ab+b^2}$$

$$S = a^2+ab+b^2 > 0$$

$$\frac{S+3}{3} \geq \sqrt[3]{S} \Rightarrow (S+3)^3 \geq 27S$$

$$\Rightarrow S^3+9S^2+27S-27S \geq 0$$

$$\Rightarrow S^3+9S^2 > 0, \text{ true, } (\forall)S > 0$$

Then

$$\int_a^b e^{x^2}\,dx \geq (b-a)\sqrt[3]{a^2+ab+b^2}$$

Solution 2 by Marin Chirciu – Romania

Using $e^t \geq t+1, t \geq 0$, for $t = x^2 \geq 0$ we obtain:

$$\int_a^b e^{x^2}\,dx \geq \int_a^b (x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)\Bigg|_a^b = \frac{b^3-a^3}{3}+b-a =$$

$$= (b-a)\left(\frac{b^2+ba+a^2}{3}+1\right) \overset{(1)}{\geq} (b-a)\sqrt[3]{a^2+ab+b^2}$$

where (1) $\Leftrightarrow \frac{b^2+ba+a^2}{3}+1 \geq \sqrt[3]{a^2+ab+b^2}$, which follows from:

We denote $\sqrt[3]{a^2+ab+b^2}=t>0$ and the inequality $\frac{b^2+ba+a^2}{3}+1\geq\sqrt[3]{a^3+ab+b^2}$ can be written:

$$\frac{t^3}{3}+1\geq 1 \Leftrightarrow t^3-3t+3\geq 0, \text{ true from } t^3-3t+3\overset{(2)}{\geq}1>0,$$

where (2) $\Leftrightarrow t^3-3t+3\geq 1\Leftrightarrow t^3-3t+2\geq 0\Leftrightarrow (t-1)^2(t+2)\geq 0$ with equality for $t=1$.

Equality holds if and only if $a=b$.

Solution 3 by Hikmat Mammadov – Azerbaijan

The function $f: x\to e^{x^2}$ is convex so $\forall x\in[a,b]$

$$f(x)\geq f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)$$

So (by integrations on $[a,b]$)

$$\int_a^b f(x)\,dx\geq (b-a)f\left(\frac{a+b}{2}\right)$$

The derivative function of $g: t\to e^{\frac{3t}{4}}-t$ is $g': t\to \frac{3}{4}e^{\frac{3t}{4}}-1$

So the minimum of the function g is $g\left(\frac{4}{3}\ln\left(\frac{4}{3}\right)\right)=\frac{4}{3}\left(1-\ln\left(\frac{4}{3}\right)\right)\geq 0$

So $\forall t\in\mathbb{R}$ and $g(t)\geq 0$

With $t=(a+b)^2$ and we get $e^{\frac{3}{4}(a+b)^2}\geq (a+b)^2$

Since $(a+b)^2\geq a^2+b^2+ab$ and we get $e^{3\left(\frac{a+b}{2}\right)^2}\geq a^2+b^2+ab$

So $e^{\left(\frac{a+b}{2}\right)^2}\geq\sqrt[3]{a^2+b^2+ab}$

That gives $(b-a)f\left(\frac{a+b}{2}\right)\geq (b-a)\sqrt[3]{a^2+b^2+ab}$

And finally:

$$\int_a^b e^{x^2}\,dx\geq (b-a)\sqrt[3]{a^2+b^2+ab}$$