ROMANIAN MATHEMATICAL MAGAZINE

UP.540 If $a,b\in\mathbb{R}, a<b, f:[a,b]\to(0,\infty), f-$ continuous then:

$$3\int_a^b f(x)\,dx+\frac{1}{(b-a)^2}\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge 4(b-a)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

For $x,y,z\in[a,b], f(x),f(y),f(z)>0$

$$f(x)+f(y)+f(z)+\frac{1}{f(x)\cdot f(y)\cdot f(z)}\overset{AM-GM}{\ge}4\sqrt[4]{f(x)f(y)f(z)\cdot\frac{1}{f(x)f(y)f(z)}}=4$$

$$\int_a^b\int_a^b\int_a^b\left(f(x)+f(y)+f(z)+\frac{1}{f(x)\cdot f(y)\cdot f(z)}\right)dxdydz\ge\int_a^b\int_a^b\int_a^b 4dxdydz$$

$$3\int_a^b f(x)dx\cdot(b-a)^2+\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge 4(b-a)^3$$

$$3\int_a^b f(x)\,dx+\frac{1}{(b-a)^2}\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge 4(b-a)$$

Solution 2 by Marin Chirciu – Romania

With CBS inequality we have:

$$\int_a^b f(x)\,dx\cdot\int_a^b\frac{1}{f(x)}dx\ge\left(\int_a^b 1\,dx\right)^2=(b-a)^2\Rightarrow$$

$$\Rightarrow\int_a^b f(x)\,dx\cdot\int_a^b\frac{1}{f(x)}dx\ge(b-a)^2\Rightarrow$$

$$\Rightarrow\int_a^b f(x)\,dx\ge\frac{(b-a)^2}{\int_a^b\frac{1}{f(x)}dx}\quad(1)$$

Using (1) it suffices to prove that:

$$3\cdot\frac{(b-a)^2}{\int_a^b\frac{1}{f(x)}dx}+\frac{1}{(b-a)^2}\left(\int_a^b\frac{1}{f(x)}dx\right)^3\ge 4(b-a)$$

which follows from:

We denote $\int_a^b \frac{1}{f(x)}dx = I$ and the above inequality can be written

$$3\cdot\frac{(b-a)^2}{I}+\frac{1}{(b-a)^2}I^3 \geq 4(b-a) \Leftrightarrow$$

$$\Leftrightarrow I^4-4(b-a)^3I+3(b-a)^4 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow [I-(b-a)^2]^2[I^2+2(b-a)I+3(b-a)^2] \geq 0$$

with equality for $I = b-a$.