A SIMPLE PROOF FOR EVERITT'S INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper we will give a simple proof for Everitt's inequality and a few applications.

EVERITT'S INEQUALITY

If $0 < x \leq \pi$ then:

$$\frac{\sin x}{x} < \min\left\{\frac{2(1-\cos x)}{x^2}, \frac{2+\cos x}{3}\right\} \tag{1}$$

Proof.

$$\frac{\sin x}{x} < \frac{2(1-\cos x)}{x^2} \tag{2}$$

$$x\sin x < 2 - 2\cos x$$

$$x\sin x + 2\cos x < 2$$

$$\text{Let be } f : (0,\pi] \to \mathbb{R}; f(x) = x\sin x + 2\cos x$$

$$f'(x) = \sin x + x\cos x - 2\sin x$$

$$f'(x) = x\cos x - \sin x$$

$$f''(x) = \cos x - x\sin x - \cos x = -x\sin x < 0$$

$$f''(x) < 0 \Rightarrow f' \text{ decreasing } \Rightarrow$$

$$\sup_{x\in(0,\pi]} f'(x) = \lim_{\substack{x\to 0\\ x>0}} f'(x) = \lim_{\substack{x\to 0\\ x>0}} (x\cos x - \sin x) = 0$$

$$f'(x) < 0 \Rightarrow f \text{ decreasing } \Rightarrow$$

$$\sup f(x) = \lim_{\substack{x\to 0\\ x>0}} f(x) = \lim_{\substack{x\to 0\\ x>0}} (x\sin x + 2\cos x) = 2$$

$$\Rightarrow f(x) < 0; (\forall)x \in (0,\pi] \Rightarrow (2).$$

$$\frac{\sin x}{x} < \frac{2+\cos x}{3} \tag{3}$$

$$3\sin x < 2x + x\cos x$$

$$3\sin x - 2x - x\cos x < 0$$

$$\text{Let be } g : (0,\pi] \to \mathbb{R}; g(x) = 3\sin x - 2x - x\cos x$$

$$g'(x) = 3\cos x - 2 - \cos x + x\sin x$$

$$g'(x) = 2\cos x - 2 + x\sin x$$

$$g''(x) = -2\sin x + \sin x + x\cos x$$

$$g''(x) = -\sin x + x\cos x$$

$$g'''(x) = -\cos x + \cos x - x\sin x$$

$$g'''(x) = -x\sin x < 0; (\forall)x \in (0,\pi]$$

$$g'''(x) < 0 \Rightarrow g''(x) \text{ - decreasing } \Rightarrow$$
$$\sup_{x\in(0,\pi]} g''(x) = \lim_{\substack{x\to 0\\ x>0}} g''(x) = \lim_{\substack{x\to 0\\ x>0}} (-\sin x + x\cos x) = 0$$
$$\Rightarrow g''(x) < 0 \Rightarrow g'(x) \text{ - decreasing } \Rightarrow$$
$$\sup_{x\in(0,\pi]} g'(x) = \lim_{\substack{x\to 0\\ x>0}} g'(x) = \lim_{\substack{x\to 0\\ x>0}} (2\cos x - 2 + x\sin x) = 0$$
$$\Rightarrow g'(x) < 0 \Rightarrow g(x) \text{ - decreasing } \Rightarrow$$
$$\Rightarrow g(x) < 0; (\forall) x \in (0,\pi] \Rightarrow (3)$$

By $(2); (3) \Rightarrow (1)$. □

Corollary.
If $x \in [0,\pi)$ then:

$$\frac{\sin x}{\pi - x} < \min\left\{\frac{2(1+\cos x)}{(\pi-x)^2}; \frac{2-\cos x}{3}\right\}$$

Proof.
We replace x in (1) with $\pi - x$:

$$0 < x \le \pi \Rightarrow 0 > -x \ge -\pi \Rightarrow \pi > \pi - x \ge 0$$
$$\Rightarrow \frac{\sin(\pi-x)}{\pi-x} < \min\left\{\frac{2(1-\cos(\pi-x))}{(\pi-x)^2}; \frac{2+\cos(\pi-x)}{3}\right\}$$
$$\frac{\sin x}{\pi - x} < \min\left\{\frac{2(1+\cos x)}{(\pi-x)^2}; \frac{2-\cos x}{3}\right\}$$

□

Application.
If $0 < a \le b \le \pi$ then:

$$\int_a^b \frac{\sin x}{x}dx + 2\int_a^b \frac{\cos x}{x^2}dx \le \frac{2(b-a)}{ab}$$

Proof.
We will use (2):

$$\frac{\sin x}{x} < \frac{2(1-\cos x)}{x^2}$$
$$\int_a^b \frac{\sin x}{x}dx \le \int_a^b \frac{2-2\cos x}{x^2}dx$$
$$\int_a^b \frac{\sin x}{x}dx \le 2\int_a^b \frac{1}{x^2}dx - 2\int_a^b \frac{\cos x}{x^2}dx$$
$$\int_a^b \frac{\sin x}{x}dx + 2\int_a^b \frac{\cos x}{x^2}dx \le -2\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{2(b-a)}{ab}$$

Equality holds for $a = b$.

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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