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1301. If $t_1, t_2, \dots, t_n > 0$ such that $t_1 + t_2 + \dots + t_n = 1$

Then prove that:

$$\frac{(1-t_1)\cdot(1-t_2)\cdot\ldots\cdot(1-t_n)}{t_1\cdot t_2\cdot\ldots\cdot t_n} \ge (n-1)^n$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by Khanh Hai-Vietnam

$$1 - t_1 = t_2 + t_3 + \dots + t_n \ge (n-1)\sqrt[n-1]{t_2 t_3 \dots t_n} \quad \text{(AM-GM)}$$

Similarly, we can prove that:

$$1 - t_2 \ge (n-1)\sqrt[n-1]{t_1 t_3 \dots t_n}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$1 - t_n \ge n(n-1)\sqrt[n-1]{t_1 t_2 \dots t_{n-1}}$$

$$\Rightarrow \prod(1-t_1) \ge (n-1)^n \cdot t_1 t_2 \dots t_n \Rightarrow \frac{\prod(1-t_1)}{t_1 t_2 \dots t_n} \ge (n-1)^n \quad \text{Q.E.D.}$$

Solution 2 by Ali Babazadeh-Azerbaijan

$$t_1 + t_2 + \dots + t_n = 1$$

$$S = \frac{(1-t_1)(1-t_2)\dots(1-t_n)}{t_1\cdot t_2\cdot\ldots\cdot t_n} \ge (n-1)^n$$

a) $(1-t_1)(1-t_2)\dots(1-t_n) = (t_2+\dots+t_n)(t_1+\dots+t_n)\dots(t_1+\dots+t_{n-1}) \ge$

$$\ge (n-1)\sqrt[n-1]{t_2\cdot\ldots\cdot t_n}\cdot(n-1)\sqrt[n-1]{t_1\cdot\ldots\cdot t_n}\cdot\ldots\cdot(n-1)\sqrt[n-1]{t_1\cdot\ldots\cdot t_{n-1}} =$$

$$= (n-1)^n\sqrt[n-1]{(t_1\cdot t_2\cdot\ldots\cdot t_n)^{n-1}} = (n-1)^n(t_1\cdot t_2\cdot\ldots\cdot t_n)$$

Then: $S \ge (n-1)^n$

1302. If $a, b.c \ge 0$ such that $a + b + c = 2$ then

$$a^2b^2 + b^2c^2 + c^2a^2 + \frac{11}{8}abc \le 1$$

Proposed by Tran Quoc Thinh-Vietnam

Solution 1 by Nguyen Van Canh-Ben Tre-Vietnam

Let us denote $q = ab + bc + ca \le \frac{(a+b+c)^2}{3} = \frac{4}{3}, r = abc$. We have

$$(a-b)^2(b-c)^2(c-a)^2 \ge 0 \Leftrightarrow 4q^2 - 4q^3 + 4(9q-8)r - 27r^2 \ge 0$$

$\Leftrightarrow r \ge \frac{2(9q-8)-2(4-3q)\sqrt{4-3q}}{27}$. Now,

$$a^2b^2+b^2c^2+c^2a^2+\frac{11}{8}abc\le 1;$$
$$\Leftrightarrow q^2-4r+\frac{11}{8}r\le 1;$$
$$\Leftrightarrow \frac{21}{8}r+1-q^2\ge 0;\ (*)$$

- If $\boxed{0<q\le 1}$ then $\frac{21}{8}r+1-q^2\overset{r\ge 0}{\ge}\frac{21}{8}.0+0=0\Rightarrow (*)$ true.

Equality $\Leftrightarrow\begin{cases}ab+bc+ca=1\\ a+b+c=2\\ abc=0\end{cases}\Leftrightarrow (a,b,c)\in\{(1,1,0),(0,1,1),(1,0,1)\}.$

- If $\boxed{1<q\le\frac{4}{3}}$ then we need to prove that

$$\frac{21}{8}\left(\frac{2(9q-8)-2(4-3q)\sqrt{4-3q}}{27}\right)-q^2+1\ge 0;$$
$$\Leftrightarrow 63q-56+36(1-q^2)\ge 7(4-3q)\sqrt{4-3q};$$
$$\Leftrightarrow (12q-5)(4-3q)\ge 7(4-3q)\sqrt{4-3q};$$
$$\Leftrightarrow \underbrace{(4-3q)}_{\ge 0}\left(12q-5-7\sqrt{4-3q}\right)\ge 0;$$

We just check: $12q-5-7\sqrt{4-3q}>0;$
$$\Leftrightarrow (12q-5)^2>49(4-3q);$$
$$\Leftrightarrow 144q^2+27q-171>0;$$
$$\Leftrightarrow 9(q-1)(16q+19)>0\ \text{(true)}.$$

$\Rightarrow (*)$ true. Equality $\Leftrightarrow\begin{cases}q=\frac{4}{3}\\ p=2\\ r=\frac{8}{27}\end{cases}\Leftrightarrow a=b=c=\frac{2}{3}$. Proved.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Homogenizing the given inequality we get the equivalent expression,
$$16(a^2b^2+b^2c^2+c^2a^2)+11abc(a+b+c)\le (a+b+c)^4.$$
which, after expanding becomes,
$$10(a^2b^2+b^2c^2+c^2a^2)\le a^4+b^4+c^4+abc(a+b+c)+4\sum_{cyc}(a^3b+ab^3).$$
By AM − GM inequality, we have
$$10(a^2b^2+b^2c^2+c^2a^2)=5\sum_{cyc}2a^2b^2\le 5\sum_{cyc}(a^3b+ab^3).$$
So it suffices to prove that
$$\sum_{cyc}(a^3b+ab^3)\le a^4+b^4+c^4+abc(a+b+c),$$
which is equivalent to
$$0\le a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b),$$

which is Schur's inequality. So the proof is complete.

Equality holds iff $a = b = c = \frac{2}{3}$ or $a = b = 1, c = 0$ and permutations.

1303. If $m, n \in \mathbb{N}, n \geq 2, a, b, c > 0, (abc)^m + a^m b^m + b^m c^m + c^m a^m = 4$ then:

$$\sqrt[n]{a^m + 2} + \sqrt[n]{b^m + 2} + \sqrt[n]{c^m + 2} \geq 3\sqrt[n]{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Tapas Das-India

$$(abc)^m + a^m b^m + b^m c^m + c^m a^m = 4$$

The given condition can be rewritten as follows:

$$\frac{1}{a^m + 2} + \frac{1}{b^m + 2} + \frac{1}{c^m + 2} = 1$$

By Holder's inequality, we have:

$$\left(\sum_{cyc} \sqrt[n]{a^m + 2}\right)^n \left(\sum \frac{1}{a^m + 2}\right) \geq (1 + 1 + 1)^{n+1} \Rightarrow \left(\sum_{cyc} \sqrt[n]{a^m + 2}\right)^n \cdot 1 \geq 3^{n+1}$$

$$\therefore \sum_{cyc} \sqrt[n]{a^m + 2} \geq 3\sqrt[n]{3}$$

Solution 2 by Sidi Abdellah Lemrabott-Mauritania

Pose: $a^m = x, b^m = y, c^m = z \Leftrightarrow xyz + xy + yz + zx = 4$

Now we need to prove that: $\sqrt[n]{x + 2} + \sqrt[n]{y + 2} + \sqrt[n]{z + 2} \geq 3\sqrt[n]{3}$

Pose: $x + 2 = a', y + 2 = b', z + 2 = c'$ then the condition

$xyz + xy + yz + zx = 4$ becomes:

$$a'b'c' = a'b' + b'c' + c'a' \overset{AM-GM}{\geq} 3\sqrt[3]{(a'b'c')^2} \Leftrightarrow a'b'c' \geq 27$$

$$\sqrt[n]{a'} + \sqrt[n]{b'} + \sqrt[n]{c'} \geq 3\sqrt[3]{\sqrt[n]{a'b'c'}} \geq 3\sqrt[3]{\sqrt[n]{27}} = 3\sqrt[n]{3} \quad \text{(Q.E.D.)}$$

Equality holds if: $(a' = b' = c' = 3 \Leftrightarrow x = y = z = 1 \Leftrightarrow a = b = c = 1)$

Solution 3 by Khanh Hai-Vietnam

We have: $(abc)^m + \sum (ab)^m = 4$

$$\Leftrightarrow (abc)^m + 2\sum(ab)^m + 4\sum a^m + 8 = \sum(ab)^m + 4\sum a^m + 12$$

$$\Leftrightarrow \prod(a^m+2) = \sum(a^m+2)(b^m+2) \Rightarrow \sum\frac{1}{a^m+2} = 1$$

By Holder's inequality, we have:

$$\left(\sum\sqrt[n]{a^m+2}\right)^n\cdot\sum\frac{1}{a^m+2} \ge 3^{n+1} \Leftrightarrow \left(\sum\sqrt[n]{a^m+2}\right)^n \ge 3^{n+1} \Leftrightarrow \sum\sqrt[n]{a^m+2} \ge 3\sqrt[n]{3}$$

1304. Let $a, b, c \ge 0$. Prove that :

$$\sum_{cyc} a^3 + 3abc \le \sum_{cyc} ab(a+b) + \frac{2023}{2022}\cdot\frac{\sum_{cyc} a^4}{\sum_{cyc} a^2b}.(|a-b|+|b-c|+|c-a|)^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^2b \ne 0 \text{ and } a,b,c \ge 0 \therefore a^2b+b^2c+c^2a > 0$$

$$\therefore \text{maximum 1 variable can be} = 0$$

Case 1 Exactly 1 variable $= 0$ and WLOG we may assume $a = 0$ (with $b, c > 0$)

$$\text{and then : } \sum_{cyc} a^3 + 3abc \le \sum_{cyc} ab(a+b)$$

$$+\frac{2023}{2022}\cdot\frac{\sum_{cyc} a^4}{\sum_{cyc} a^2b}.(|a-b|+|b-c|+|c-a|)^2$$

$$\Leftrightarrow b^3+c^3 \overset{(1)}{\le} bc(b+c) + \frac{b^4+c^4}{b^2c}(b+c+|b-c|)^2$$

$$\text{We have : } bc(b+c) + \frac{b^4+c^4}{b^2c}(b+c+|b-c|)^2$$

$$= bc(b+c) + \frac{b^4+c^4}{b^2c}\left((b+c)^2+(b-c)^2+2(b+c)|b-c|\right)$$

$$\ge bc(b+c) + \frac{2(b^4+c^4)(b^2+c^2)}{b^2c} \overset{CBS}{\ge} bc(b+c) + \frac{2(b^3+c^3)^2}{b^2c}$$

$$\ge bc(b+c) + \frac{2(b^3+c^3).bc(b+c)}{b^2c} = bc(b+c) + 2(b^3+c^3)\left(\frac{b^2c+bc^2}{b^2c}\right)$$

$$= bc(b+c) + 2(b^3+c^3)\left(1+\frac{c}{b}\right) = bc(b+c) + 2(b^3+c^3) + 2(b^3+c^3).\frac{c}{b}$$

$$> b^3+c^3 \Rightarrow (1) \text{ is true (strict inequality)}$$

Case 2 $a, b, c > 0$

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1)$

$$\Rightarrow a = s - x, b = s - y, c = s - z \text{ and such substitutions } \Rightarrow$$

$$\sum_{cyc} ab = \sum_{cyc} (s - x)(s - y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (2) \text{ and}$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (2)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3) \text{ and}$$

$$\sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a + b)(b + c)(c + a) \overset{\text{via (1)}}{=} s^3 - 3xyz$$

$$\Rightarrow \sum_{cyc} a^3 = s^3 - 12Rrs \to (4) \text{ and also}, \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1) and (2)}}{=} (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (5)$$

- $$\sum_{cyc} a^3 + 3abc - \sum_{cyc} ab(a + b) \overset{\text{via (4)}}{=} s^3 - 12Rrs + 3\prod_{cyc}(s - x)$$

$$-\sum_{cyc}(s - x)(s - y)z = s^3 - 12Rrs + 3r^2s - \sum_{cyc}(-s^2 + sz + xy)z$$

$$= s^3 - 12Rrs + 3r^2s + s^2.2s - s.2(s^2 - 4Rr - r^2) - 12Rrs$$

$$\therefore \sum_{cyc} a^3 + 3abc - \sum_{cyc} ab(a + b) \overset{(\bullet)}{=} s(s^2 - 16Rr + 5r^2)$$

- $$\sum_{cyc} a^2b = \sum_{cyc}\left(a^2\left(\sum_{cyc} a - c - a\right)\right) = \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - \sum_{cyc} a^3 - \sum_{cyc}\frac{a^2c^2}{c}$$

$$\overset{\text{via (1),(3) and (4) and Bergstrom}}{\leq} s(s^2 - 8Rr - 2r^2) - s^3 + 12Rrs - \frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} a}$$

$$\overset{\text{via (1) and (2)}}{=} s(s^2 - 8Rr - 2r^2) - s^3 + 12Rrs - \frac{(4Rr + r^2)^2}{s}$$

$$= \frac{r}{s}.\left((4R - 2r)s^2 - r(4R + r)^2\right) \Rightarrow \frac{2023}{2022}.\frac{\sum_{cyc} a^4}{\sum_{cyc} a^2b}.(|a - b| + |b - c| + |c - a|)^2$$

$$\geq \frac{\left(\sum_{cyc} a^2\right)^2 - 2\sum_{cyc} a^2b^2}{\frac{r}{s}.\left((4R - 2r)s^2 - r(4R + r)^2\right)}.\left(\sum_{cyc}(y - z)^2 + 2\sum_{cyc}|y - z||z - x|\right)$$

$$\overset{\text{via (3) and (5)}}{=}\frac{s\left(\left(s^2-8Rr-2r^2\right)^2-2r^2\left((4R+r)^2-2s^2\right)\right)}{r\left((4R-2r)s^2-r(4R+r)^2\right)}.\left(2\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)+2\sum_{cyc}|y-z||z-x|\right)$$

$$\geq\frac{s\left(\left(s^2-8Rr-2r^2\right)^2-2r^2\left((4R+r)^2-2s^2\right)\right)}{r\left((4R-2r)s^2-r(4R+r)^2\right)}.\left(2\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)\right)$$

$$=\frac{2s\left(\left(s^2-8Rr-2r^2\right)^2-2r^2\left((4R+r)^2-2s^2\right)\right)\left(s^2-12Rr-3r^2\right)}{r\left((4R-2r)s^2-r(4R+r)^2\right)}$$

$$\overset{?}{\geq}s\left(s^2-16Rr+5r^2\right)\Leftrightarrow 2s^6-\left(60Rr+4r^2\right)s^4+r^2s^2\left(528R^2+84Rr+15r^2\right)$$

$$-r^3\left(1024R^3+624R^2r+120Rr^2+7r^3\right)\underset{(*)}{\overset{?}{\geq}}0 \text{ and } \because 2\left(s^2-16Rr+5r^2\right)^3$$

$$\overset{\text{Gerretsen}}{\geq}0\ \therefore \text{ in order to prove } (*), \text{ it suffices to prove :}$$

$$\text{LHS of } (*)\geq 2\left(s^2-16Rr+5r^2\right)^3$$

$$\Leftrightarrow (36R-34r)s^4-rs^2\left(1008R^2-1044Rr+135r^2\right)$$

$$+r^2\left(7168R^3-8304R^2r+2280Rr^2-257r^3\right)\underset{(**)}{\overset{?}{\geq}}0$$

$$\text{and } \because (36R-34r)\left(s^2-16Rr+5r^2\right)^2\overset{\text{Gerretsen + Euler}}{\geq}0\ \therefore \text{ in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**)\geq(36R-34r)\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow\left(144R^2-404Rr+205r^2\right)s^2\underset{(***)}{\overset{?}{\geq}}r\left(2048R^3-6160R^2r+4060Rr^2-593r^3\right)$$

$\boxed{\text{Case 2i}}$ $144R^2-404Rr+205r^2\geq 0$ $\left(\Leftrightarrow t=\frac{R}{r}\geq 2.1405 \text{ (approximately)}\right)$

$$\text{and then : LHS of } (***)\overset{\text{Gerretsen}}{\geq}\left(144R^2-404Rr+205r^2\right)\left(16Rr-5r^2\right)$$

$$\overset{?}{\geq}r\left(2048R^3-6160R^2r+4060Rr^2-593r^3\right)\Leftrightarrow 32t^3-128t^2+155t-54\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)(32t(t-2)+27)\overset{?}{\geq}0\rightarrow \text{true} \because t>2\Rightarrow (***) \text{ is true (strict inequality)}$$

$\boxed{\text{Case 2ii}}$ $144R^2-404Rr+205r^2<0$ and then : LHS of $(***)\overset{\text{Gerretsen}}{\geq}$

$$-\left(-\left(144R^2-404Rr+205r^2\right)\right)\left(4R^2+4Rr+3r^2\right)\overset{?}{\geq}$$

$$r\left(2048R^3-6160R^2r+4060Rr^2-593r^3\right)$$

$$\Leftrightarrow 144t^4-772t^3+1449t^2-1113t+302\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\left((t-2)(144t(t-2)+92t+89)+27\right)\overset{?}{\geq}0\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow (***)$$

is true $\therefore$ combining cases 2i, 2ii, $(***)\Rightarrow(**)\Rightarrow(*)$ is true $\forall$ triangles with

sides $(b+c),(c+a),(a+b)$ $\therefore \frac{2023}{2022}.\frac{\sum_{cyc}a^4}{\sum_{cyc}a^2b}.\left(|a-b|+|b-c|+|c-a|\right)^2$

$$\geq s(s^2-16Rr+5r^2)\overset{via\,(\bullet)}{=}\sum_{cyc}a^3+3abc-\sum_{cyc}ab(a+b)\therefore\forall\, a,b,c>0,$$

$$\sum_{cyc}a^3+3abc\leq\sum_{cyc}ab(a+b)+\frac{2023}{2022}\cdot\frac{\sum_{cyc}a^4}{\sum_{cyc}a^2b}\cdot(|a-b|+|b-c|+|c-a|)^2$$

Hence, combining cases 1 and 2, $\sum_{cyc}a^3+3abc\leq\sum_{cyc}ab(a+b)$

$$+\frac{2023}{2022}\cdot\frac{\sum_{cyc}a^4}{\sum_{cyc}a^2b}\cdot(|a-b|+|b-c|+|c-a|)^2\ \forall\, a,b,c\geq 0,$$

$''=''$ iff $a=b=c>0$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Chebyshev and AM − GM inequalities, we have

$$a^4+b^4+c^4\geq\frac{1}{3}(a+b+c)(a^3+b^3+c^3)$$

$$=\frac{1}{3}(a+b+c)\left(\frac{2a^3+b^3}{3}+\frac{2b^3+c^3}{3}+\frac{2c^3+a^3}{3}\right)$$

$$\geq\frac{1}{3}(a+b+c)(a^2b+b^2c+c^2a).$$

By the triangle inequality, we have

$$(|a-b|+|b-c|+|c-a|)^2=\sum_{cyc}(a-b)^2+\sum_{cyc}|a-b|(|b-c|+|c-a|)$$

$$\geq\sum_{cyc}(a-b)^2+\sum_{cyc}(a-b)^2=4(a^2+b^2+c^2-ab-bc-ca).$$

From these results and since $\frac{2023}{2022},\frac{4}{3}>1$, then we have

$$RHS_{(*)}\geq\sum_{cyc}ab(a+b)+(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$=\sum_{cyc}a^3+3abc+\sum_{cyc}a(b-c)^2\geq\sum_{cyc}a^3+3abc=LHS_{(*)},$$

as desired. Equality holds iff $a=b=c$.

1305. Let $a,b,c>0$. *Prove that*:

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)$$

$$\leq\frac{9}{4}+\frac{2023^2(a^2+b^2+c^2)}{abc}(|a-b|+|b-c|+|c-a|)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0,$$
$$y+z-x=2a>0 \text{ and } z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$$
$$\Rightarrow x,y,z \text{ form sides of a triangle with semiperimeter, circumradius and inradius}$$
$$= s,R,r \text{ (say) yielding } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x,$$
$$b = s-y, c = s-z \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$
$$\Rightarrow \sum_{cyc} ab = 4Rr+r^2 \to (2) \text{ and } \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab$$
$$\overset{\text{via (1) and (2)}}{=} s^2 - 2(4Rr+r^2) \Rightarrow \sum_{cyc} a^2 = s^2-8Rr-2r^2 \to (3)$$
$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)-\frac{9}{4} \overset{\text{via (2)}}{=} (4Rr+r^2)\sum_{cyc}\frac{1}{x^2}-\frac{9}{4}$$
$$= \frac{(4Rr+r^2)}{16R^2r^2s^2}\left(\sum_{cyc} x^2y^2\right)-\frac{9}{4} \overset{\text{Goldstone}}{\leq} \frac{(4Rr+r^2)(4R^2s^2)}{16R^2r^2s^2}-\frac{9}{4}$$
$$\therefore (ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)-\frac{9}{4} \overset{(\bullet)}{\leq} \frac{R-2r}{r}$$
$$\frac{a^2+b^2+c^2}{abc}(|a-b|+|b-c|+|c-a|) \overset{\text{via (3)}}{=} \frac{s^2-8Rr-2r^2}{\prod_{cyc}(s-x)}.\sum_{cyc}|y-z|$$
$$= \frac{s^2-8Rr-2r^2}{r^2s}.\sqrt{\sum_{cyc}(y-z)^2+2\sum_{cyc}|y-z||z-x|}$$
$$\geq \frac{s^2-8Rr-2r^2}{r^2s}.\sqrt{2\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)}$$
$$= \frac{s^2-8Rr-2r^2}{r^2s}.\sqrt{2(s^2-12Rr-3r^2)} \overset{?}{\geq} \frac{R-2r}{r}$$
$$\Leftrightarrow 2(s^2-12Rr-3r^2)(s^2-8Rr-2r^2)^2 \underbrace{\overset{?}{\geq}}_{(*)} (R-2r)^2r^2s^2$$
$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} 2(4Rr-8r^2)(s^2-8Rr-2r^2)^2$$
$$= 8r(R-2r)(s^2-8Rr-2r^2)^2 \text{ and } \because R-2r \overset{\text{Euler}}{\geq} 0$$
$$\therefore \text{in order to prove } (*), \text{it suffices to prove : } 8(s^2-8Rr-2r^2)^2 \overset{(**)}{>} r(R-2r)s^2$$
$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\geq} 8(8Rr-7r^2)(s^2-8Rr-2r^2) \overset{?}{>} r(R-2r)s^2$$
$$\Leftrightarrow (63R-54r)s^2 \underbrace{\overset{?}{>}}_{(***)} r(512R^2-320Rr-112r^2)$$
$$\text{Once again, LHS of } (***) \overset{\text{Gerretsen}}{\geq} (63R-54r)(16Rr-5r^2) \overset{?}{>}$$

$$r(512R^2-320Rr-112r^2)\Leftrightarrow 496R^2-859Rr+382r^2\overset{?}{>}0$$

$$\Leftrightarrow (R-2r)(496R+133r)+648r^2\overset{?}{>}0\to \text{true} \because R\overset{\text{Euler}}{\geq}2r$$

$$\Rightarrow (***)\Rightarrow (**)\Rightarrow (*)\text{ is true} \therefore \frac{a^2+b^2+c^2}{abc}(|a-b|+|b-c|+|c-a|)\geq\frac{R-2r}{r}$$

$$\overset{\text{via }(\bullet)}{\geq}(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)-\frac{9}{4}$$

$$\Rightarrow (ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)-\frac{9}{4}$$

$$\leq\frac{a^2+b^2+c^2}{abc}(|a-b|+|b-c|+|c-a|)$$

$$\leq\frac{2023^2(a^2+b^2+c^2)}{abc}(|a-b|+|b-c|+|c-a|)$$

$$\Rightarrow (ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)$$

$$\leq\frac{9}{4}+\frac{2023^2(a^2+b^2+c^2)}{abc}(|a-b|+|b-c|+|c-a|)\ \forall\, a,b,c>0,$$

$$''=''\ \text{iff } a=b=c\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\overset{AM-GM}{\leq}$$

$$\leq (ab+bc+ca)\left(\frac{1}{4ab}+\frac{1}{4bc}+\frac{1}{4ca}\right)$$

$$=\frac{9}{4}+\frac{(ab+bc+ca)(a+b+c)-9abc}{4abc}=\frac{9}{4}+\frac{c(a-b)^2+a(b-c)^2+b(c-a)^2}{4abc}$$

$$=\frac{9}{4}+\frac{|ca-bc||a-b|+|ab-ca||b-c|+|bc-ab||c-a|}{4abc}$$

$$\leq\frac{9}{4}+\frac{(ab+bc+ca)|a-b|+(ab+bc+ca)|b-c|+(ab+bc+ca)|c-a|}{4abc}$$

$$=\frac{9}{4}+\frac{(ab+bc+ca)(|a-b|+|b-c|+|c-a|)}{4abc}$$

$$\leq\frac{9}{4}+\frac{2023^2(a^2+b^2+c^2)}{abc}(|a-b|+|b-c|+|c-a|).$$

as desired. Equality holds iff $a=b=c$.

1306. If $a,b,c\geq 0, abc+4(ab+bc+ca)=256$ then:

$$a^3+b^3+c^3+a^2+b^2+c^2+abc\geq 304$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Lazaros Zachariadis-Greece

$$LHS=2\sum\left(\frac{a^3}{4}+\frac{b^3}{4}+16\right)+\sum a^2+abc-96$$

$$\geq 2\sum 3\sqrt[3]{\frac{(ab)^3 16}{4^2}}+\sum ab+abc-96=7\sum ab+abc-96$$

$$=4\sum ab+3\sum ab+abc-96$$

$$=256+3\sum ab-96\overset{(*)}{\geq}160+3\cdot\frac{144}{3}=160+144=304=RHS$$

$$\text{“=”}\ a=b=c=4$$

$$abc+4\sum ab\leq\left(\frac{\sum ab}{3}\right)^{\frac{3}{2}}+4\sum ab\quad (*)$$

$$\left(\frac{\sqrt{3\sum ab}}{9}-\frac{4}{3}\right)\left(16\sqrt{3\sum ab}+x+192\right)\geq 0$$

$$\frac{\sqrt{3\sum ab}}{9}\geq\frac{4}{3}\Leftrightarrow\sqrt{3\sum ab}\geq 12$$

$$3\sum ab\geq 144\Leftrightarrow\sum ab\geq\frac{144}{3}=48$$

Solution 2 by Khanh Hai-Vietnam

$$256=abc+4\sum ab\geq abc+12\sqrt[3]{a^2b^2c^2}$$

$$\Leftrightarrow x^3+12x^2-256\leq 0\qquad (x=\sqrt[3]{abc})$$

$$\Leftrightarrow (x-4)(x+8)^2\leq 0\Leftrightarrow x\leq 4\Leftrightarrow abc\leq 64\Rightarrow\sum ab\geq 48$$

$$2LHS=2\left(\sum a^3+2\sum a^2+abc\right)\geq 2\left(\sum a^3+\sum ab+256-4\sum ab\right)$$

$$=2\left(\sum a^3-3\sum ab+256\right)=2\sum a^3-6\sum ab+512$$

$$=a^3+b^3+64+b^3+c^3+64+c^3+a^3+64-6\sum ab+320$$

$$\geq 12\sum ab-6\sum ab+320$$

$$=6\sum ab+320\geq 608=2RHS\Rightarrow 2LHS\geq 2RHS\Rightarrow Q.E.D.$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$256=abc+4\sum ab\leq\left(\frac{a+b+c}{3}\right)^3+\frac{4}{3}(a+b+c)^2\sum a=x$$

$$x^3+36x^2-6912\geq 0$$

$$(x-12)(x+24)^2\geq 0$$

$$x\geq 12 \ \ (1)$$

$$abc=256-4\sum ab\geq 256-\frac{4}{3}x^2 \quad (2)$$

$$\sum a^3+\sum a^2+abc\geq 304$$

$$\sum a^3+\sum a^2+abc\geq \frac{(a+b+c)^3}{27}+\frac{(a+b+c)^2}{3}+abc\geq$$

$$\overset{(2)}{\geq}\frac{x^3}{27}+\frac{x^2}{3}+256-\frac{4}{3}x^2\overset{?}{\geq} 304$$

$$x^3-9x^2-432\geq 0$$

$$\underbrace{(x-12)}_{\geq 0}\underbrace{(x^2+3x+36)}_{>0}\geq 0 \ \ \text{(True)}$$

$$x\geq 12,\qquad a+b+c\geq 12,\qquad (a=b=c=4)$$

1307. If $a,b,c>0$, then :

$$\frac{3}{2}\cdot\sqrt[3]{\frac{3}{a^3+b^3+c^3}}\leq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\leq\frac{3}{2}\cdot\frac{a+b+c}{ab+bc+ca}$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Now,}\ \frac{3}{2}\cdot\sqrt[3]{\frac{3}{a^3+b^3+c^3}}\overset{\text{Holder}}{\leq}\frac{3}{2}\cdot\sqrt[3]{\frac{27}{\left(\sum_{\text{cyc}}a\right)^3}}=\frac{9}{2\sum_{\text{cyc}}a}$$

$$\text{and}\ \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\overset{\text{Bergstrom}}{\geq}\frac{9}{2\sum_{\text{cyc}}a}$$

$$\therefore\frac{3}{2}\cdot\sqrt[3]{\frac{3}{a^3+b^3+c^3}}\leq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{\text{cyc}}a=\sum_{\text{cyc}}x=2s\Rightarrow\sum_{\text{cyc}}a=s\rightarrow(1)\Rightarrow a=s-x, b=s-y,$ $c=s-z$ and such substitutions $\Rightarrow\sum_{\text{cyc}}ab=\sum_{\text{cyc}}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (2)$$

$$\therefore \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{3}{2} \cdot \frac{a+b+c}{ab+bc+ca} \overset{via\ (1),(2)}{\Leftrightarrow} \sum_{cyc} \frac{1}{x} \leq \frac{3s}{2(4Rr+r^2)}$$

$$\Leftrightarrow \frac{s^2+4Rr+r^2}{4Rrs} \leq \frac{3s}{2(4Rr+r^2)} \Leftrightarrow (2R-r)s^2 \geq r(4R+r)^2 \to \text{true}$$

$$\because (2R-r)s^2 - r(4R+r)^2 \overset{Gerretsen}{\geq} (2R-r)(16Rr-5r^2) - r(4R+r)^2$$

$$= 2r(8R-r)(R-2r) \overset{Euler}{\geq} 0 \therefore \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{3}{2} \cdot \frac{a+b+c}{ab+bc+ca} \text{ and so,}$$

$$\frac{3}{2} \cdot \sqrt[3]{\frac{3}{a^3+b^3+c^3}} \leq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{3}{2} \cdot \frac{a+b+c}{ab+bc+ca}$$

$$\forall a,b,c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

1308. Let $a, b, c > 0$. Prove that :

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{3}{2} + \frac{2022^2(a^2+b^2+c^2)}{a^2b+b^2c+c^2a}(|a-b| + |b-c| + |c-a|)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0$,
$y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius
$= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x$,
$b = s - y, c = s - z$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (2) \text{ and } \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab$$

$$\overset{via\ (1)\ and\ (2)}{=} s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3) \text{ and}$$

$$\sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a) \overset{via\ (1)}{=} s^3 - 3xyz$$

$$\Rightarrow \sum_{cyc} a^3 = s^3 - 12Rrs \to (4)$$

$$\sum_{cyc} \frac{a}{b+c} - \frac{3}{2} = \sum_{cyc} \frac{s-x}{x} - \frac{3}{2} = \frac{s(s^2+4Rr+r^2)}{4Rrs} - \frac{3}{2}$$

$$\Leftrightarrow \sum_{cyc} \frac{a}{b+c} - \frac{3}{2} \overset{(\bullet)}{=} \frac{s^2-14Rr+r^2}{4Rr}$$

$$a^2b+b^2c+c^2a = \sum_{cyc} a^2\left(\sum_{cyc} a - c - a\right) = \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - \sum_{cyc} a^3 - \sum_{cyc} \frac{a^2c^2}{c}$$

$$\overset{\text{via (1),(3) and (4) and Bergstrom}}{\leq} s(s^2-8Rr-2r^2) - s^3 + 12Rrs - \frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} a}$$

$$\overset{\text{via (1) and (2)}}{=} s(s^2-8Rr-2r^2) - s^3 + 12Rrs - \frac{(4Rr+r^2)^2}{s}$$

$$= \frac{r}{s}.\left((4R-2r)s^2 - r(4R+r)^2\right) \Rightarrow \frac{a^2+b^2+c^2}{a^2b+b^2c+c^2a}(|a-b|+|b-c|+|c-a|)$$

$$\overset{\text{via (3)}}{\geq} \frac{s(s^2-8Rr-2r^2)}{r\left((4R-2r)s^2-r(4R+r)^2\right)}.\sum_{cyc}|y-z|$$

$$= \frac{s(s^2-8Rr-2r^2)}{r\left((4R-2r)s^2-r(4R+r)^2\right)}.\sqrt{\sum_{cyc}(y-z)^2 + 2\sum_{cyc}|y-z||z-x|}$$

$$\geq \frac{s(s^2-8Rr-2r^2)}{r\left((4R-2r)s^2-r(4R+r)^2\right)}.\sqrt{2\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right)}$$

$$= \frac{s(s^2-8Rr-2r^2)}{r\left((4R-2r)s^2-r(4R+r)^2\right)}.\sqrt{2(s^2-12Rr-3r^2)} \overset{?}{\geq} \frac{s^2-14Rr+r^2}{4Rr}$$

$$\Leftrightarrow 32R^2s^2(s^2-12Rr-3r^2)(s^2-8Rr-2r^2)^2$$

$$\underset{(*)}{\overset{?}{\geq}} \left((4R-2r)s^2 - r(4R+r)^2\right)^2(s^2-14Rr+r^2)^2$$

$$\text{Now, } s^2-12Rr-3r^2 = s^2-14Rr+r^2+2r(R-2r) \overset{\text{Euler}}{\geq} s^2-14Rr+r^2$$

$$\Rightarrow \text{LHS of } (*) \geq 32R^2s^2(s^2-14Rr+r^2)(s^2-8Rr-2r^2)^2$$

$$\overset{?}{\geq} \left((4R-2r)s^2-r(4R+r)^2\right)^2(s^2-14Rr+r^2)^2$$

$$\Leftrightarrow (s^2-14Rr+r^2)\Big(32R^2s^2(s^2-8Rr-2r^2)^2$$

$$-(s^2-14Rr+r^2)\left((4R-2r)s^2-r(4R+r)^2\right)^2\Big)$$

$$\overset{?}{\geq} 0, \text{to prove which, it suffices to prove:}$$

$$32R^2s^2(s^2-8Rr-2r^2)^2 \overset{(**)}{>} (s^2-14Rr+r^2)\left((4R-2r)s^2-r(4R+r)^2\right)^2$$

$$\left(\because s^2-14Rr+r^2 = s^2-16Rr+5r^2+2r(R-2r) \overset{\text{Gerretsen + Euler}}{\geq} 0\right)$$

$$\text{Again, LHS of } (**) = 32R^2s^2(s^2-8Rr-2r^2)(s^2-14Rr+r^2+6r(R-2r)+9r^2)$$

$$\overset{\text{Euler}}{>} 32R^2s^2(s^2-8Rr-2r^2)(s^2-14Rr+r^2)$$

$$\overset{?}{>} (s^2-14Rr+r^2)\left((4R-2r)s^2-r(4R+r)^2\right)^2$$

$$\Leftrightarrow 32R^2s^2(s^2-8Rr-2r^2)\overset{?}{>}\left((4R-2r)s^2-r(4R+r)^2\right)^2$$

$$\Leftrightarrow (16R^2+16Rr-4r^2)s^4-rs^2(128R^3+64R^2r+24Rr^2+4r^3)$$

$$-r^2(4R+r)^4\underbrace{\overset{?}{>}}_{(***)}0$$

$$\text{Now, LHS of }(***)\overset{Gerretsen}{\geq}(16R^2+16Rr-4r^2)(16Rr-5r^2)s^2$$

$$-rs^2(128R^3+64R^2r+24Rr^2+4r^3)-r^2(4R+r)^4\overset{?}{>}0$$

$$\Leftrightarrow (128R^3+112R^2r-168Rr^2+16r^3)s^2\underbrace{\overset{?}{>}}_{(****)}r(4R+r)^4$$

$$\text{Again, LHS of }(****)\overset{Gerretsen}{\geq}(128R^3+112R^2r-168Rr^2+16r^3)(16Rr-5r^2)$$

$$\overset{?}{>}r(4R+r)^4\Leftrightarrow 1792t^4+896t^3-3344t^2+1080t-81\overset{?}{>}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1792t^3+4480t^2+5616t+12312)+24543\overset{?}{>}0\to\text{true}\because t\overset{Euler}{\geq}2$$

$$\Rightarrow(****)\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{a^2+b^2+c^2}{a^2b+b^2c+c^2a}(|a-b|+|b-c|+|c-a|)\geq$$

$$\frac{s^2-14Rr+r^2}{4Rr}\overset{via\,(\bullet)}{=}\sum_{cyc}\frac{a}{b+c}-\frac{3}{2}\Rightarrow\sum_{cyc}\frac{a}{b+c}-\frac{3}{2}$$

$$\leq\frac{a^2+b^2+c^2}{a^2b+b^2c+c^2a}(|a-b|+|b-c|+|c-a|)\leq$$

$$\frac{2022^2(a^2+b^2+c^2)}{a^2b+b^2c+c^2a}(|a-b|+|b-c|+|c-a|)$$

$$\Rightarrow\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{3}{2}+\frac{2022^2(a^2+b^2+c^2)}{a^2b+b^2c+c^2a}(|a-b|+|b-c|+|c-a|)$$

$$\forall\, a,b,c>0,\text{''}=\text{''}\text{ iff }a=b=c\text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{a}{b+c}-\frac{3}{2}=\sum_{cyc}\left(\frac{a}{b+c}-\frac{1}{2}\right)=\frac{1}{2}\sum_{cyc}\left(\frac{a-b}{b+c}-\frac{c-a}{b+c}\right)=\frac{1}{2}\sum_{cyc}\left(\frac{b-c}{c+a}-\frac{b-c}{a+b}\right)$$

$$=\frac{1}{2}\sum_{cyc}\frac{(b-c)^2}{(c+a)(a+b)}=\frac{\sum_{cyc}(b+c)(b-c)^2}{2(a+b)(b+c)(c+a)}\leq\frac{\sum_{cyc}|b^2-c^2||b-c|}{2(a^2b+b^2c+c^2a)}$$

$$\leq\frac{\sum_{cyc}(a^2+b^2+c^2)|b-c|}{2(a^2b+b^2c+c^2a)}$$

$$\leq\frac{2022^2(a^2+b^2+c^2)}{a^2b+b^2c+c^2a}(|a-b|+|b-c|+|c-a|).$$

So the proof is complete. Equality holds iff $a=b=c$.

1309. Let $a,b,c\geq 0$ such that $a+b+c=2$. Prove that :

$$①\ \text{If } 1\leq k<2,\text{ then : }(ab)^k+(bc)^k+(ca)^k\leq\frac{4}{3}$$

$$②\ \text{If } k\geq 2,\text{ then : }(ab)^k+(bc)^k+(ca)^k\leq 1$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$bc\overset{a+b+c=2}{=}b(2-a-b)=-1+2b-b^2+1-ab$$

$$\therefore bc\overset{(\bullet)}{=}1-ab-(1-b)^2$$

We shall proceed with the proof of : ① If $1\leq k<2$, then : $(ab)^k+(bc)^k+(ca)^k\leq\frac{4}{3}$

We notice that if exactly 2 variables $=0$, then LHS $=0<\frac{4}{3}$ and so we proceed with the cases when exactly 1 variable $=0$ and when all 3 variables >0

Case 1 Exactly 1 variable $=0$ and WLOG we may assume $a=0$ (with $b,c>0$)

$$bc\overset{\text{via }(\bullet)}{=}1-ab-(1-b)^2=1-(1-b)^2\leq 1\Rightarrow\ln(bc)\leq 0$$

$$\Rightarrow(k-1).\ln(bc)\leq 0\Rightarrow k.\ln(bc)\leq\ln(bc)\Rightarrow\ln(bc)^k\leq\ln(bc)\Rightarrow(bc)^k\leq bc\leq 1$$

$$\therefore(ab)^k+(bc)^k+(ca)^k\leq 1$$

Case 2 $a,b,c>0$ and then, $bc\overset{\text{via }(\bullet)}{=}1-ab-(1-b)^2<1-(1-b)^2\leq 1$

$$\Rightarrow\ln(bc)<0\Rightarrow(k-1).\ln(bc)\leq 0\Rightarrow k.\ln(bc)\leq\ln(bc)\Rightarrow\ln(bc)^k\leq\ln(bc)$$

$$\Rightarrow(bc)^k\leq bc\text{ and analogs}\Rightarrow(ab)^k+(bc)^k+(ca)^k\leq\sum_{\text{cyc}}ab\leq\frac{(a+b+c)^2}{3}=\frac{4}{3}$$

$$\therefore\text{ combining cases 1 and 2, }(ab)^k+(bc)^k+(ca)^k\leq\frac{4}{3}\ \forall\, a,b,c\geq 0\,\Big|\sum_{\text{cyc}}a=2,$$

$$''='' \text{ iff } a=b=c=\frac{2}{3};k=1$$

We shall now proceed with the proof of : ② If $k\geq 2$, then : $(ab)^k+(bc)^k+(ca)^k\leq 1$

We notice that if exactly 2 variables $=0$, then LHS $=0<1$ and so we proceed with the cases when exactly 1 variable $=0$ and when all 3 variables >0

Case 1 Exactly 1 variable $=0$ and WLOG we may assume $a=0$ (with $b,c>0$)

$$bc\overset{\text{via }(\bullet)}{=}1-ab-(1-b)^2=1-(1-b)^2\leq 1\Rightarrow\ln(bc)\leq 0\text{ and }\because k-1\geq 1>0$$

$$\therefore(k-1).\ln(bc)\leq 0\Rightarrow k.\ln(bc)\leq\ln(bc)\Rightarrow\ln(bc)^k\leq\ln(bc)\Rightarrow(bc)^k\leq bc\leq 1$$

$$\therefore\text{ for } a=0\text{ (with } b,c>0\text{), then : }(ab)^k+(bc)^k+(ca)^k\leq 1, ''='' \ b=c=1$$

Case 2 $a,b,c>0$ and then, $bc\overset{\text{via }(\bullet)}{=}1-ab-(1-b)^2<1-(1-b)^2\leq 1$

$$\Rightarrow\ln(bc)<0\Rightarrow(k-2).\ln(bc)\leq 0\Rightarrow k.\ln(bc)\leq 2\ln(bc)\Rightarrow\ln(bc)^k\leq\ln(b^2c^2)$$

$$\Rightarrow(bc)^k\leq b^2c^2\text{ and analogs}\Rightarrow(ab)^k+(bc)^k+(ca)^k\leq\sum_{\text{cyc}}a^2b^2\overset{?}{<}1$$

$$\overset{a+b+c=2}{=}\frac{(a+b+c)^4}{16}\Leftrightarrow(a+b+c)^4\underset{(*)}{\overset{?}{\geq}}16\sum_{cyc}a^2b^2$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$

and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(2)$

$$\text{and}\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{\text{via (1) and (2)}}{=}(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}a^2b^2=r^2\left((4R+r)^2-2s^2\right)\to(3)\therefore(1)\text{ and }(3)\Rightarrow(*)\Leftrightarrow$$

$$s^4>16r^2\left((4R+r)^2-2s^2\right)\Leftrightarrow s^4+32r^2s^2\overset{(**)}{>}16r^2(4R+r)^2$$

Now, LHS of $(**)\overset{\text{Gerretsen}}{\geq}(16Rr+27r^2)s^2\overset{\text{Gerretsen}}{\geq}(16Rr+27r^2)(16Rr-5r^2)$

$\overset{?}{>}16r^2(4R+r)^2\Leftrightarrow 224R\overset{?}{>}151r\to$ true $\because 224R\overset{\text{Euler}}{\geq}448r>151r$

$\Rightarrow(**)\Rightarrow(*)$ is true $\therefore(ab)^k+(bc)^k+(ca)^k<1\therefore$ combining cases 1 and 2,

$(ab)^k+(bc)^k+(ca)^k\leq 1\ \forall a,b,c\geq 0\mid\sum_{cyc}a=2,''=''$ iff

$(a=0,b=c=1)$ or $(b=0,c=a=1)$ or $(c=0,a=b=1)$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

1) We have $ab\leq\left(\frac{a+b}{2}\right)^2\leq\left(\frac{a+b+c}{2}\right)^2=1$ then $(ab)^k\leq ab$ (and analogs)

Therefore

$$(ab)^k+(bc)^k+(ca)^k\leq ab+bc+ca\leq\frac{(a+b+c)^2}{3}=\frac{4}{3}.$$

Equality holds iff $k=1$ and $a=b=c=\frac{2}{3}$.

2) WLOG, we may assume that $a\geq b\geq c$.

If $b=0$ we have $c=0$ and $(ab)^k+(bc)^k+(ca)^k=0\leq 1$.

Assume now that $b>0$. By Bernoulli's inequality, we have

$$(b+c)^k=b^k\left(1+\frac{c}{b}\right)^k\geq b^k\left(1+k.\frac{c}{b}\right)\geq b^k\left(1+2.\left(\frac{c}{b}\right)^k\right)=b^k+2c^k.$$

Then

$$(ab)^k+(bc)^k+(ca)^k\leq a^k\left(b^k+c^k+c^k\right)\leq a^k(b+c)^k\leq\left(\frac{a+(b+c)}{2}\right)^{2k}=1.$$

Equality holds iff $a=b=1, c=0$ and permutations.

1310. **Find all values of $0 < k < 1$ such that**

$$a^k b^k + b^k c^k + c^k a^k \leq 1 \text{ is true for all}$$
$$a, b, c > 0 \text{ and } a + b + c = 2.$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

There is no real $0 < k < 1$ satisfies the given condition, because for

$$a = b = c = \frac{2}{3}, \text{we have}$$

$$a^k b^k + b^k c^k + c^k a^k = 3.\left(\frac{4}{9}\right)^k > 3.\frac{4}{9} > 1, \ \ \forall k \in (0,1).$$

1311. **If $a, b, c \geq 0$ such that $a + b + c = 2$ then :**

$$a^2 b^2 + b^2 c^2 + c^2 a^2 \leq 1$$

Proposed by Tran Quoc Thinh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG we may assume that $a = max\{a, b, c\}$. By AM − GM inequality, we have

$$a^2 b^2 + b^2 c^2 + c^2 a^2 = a^2(b+c)^2 - bc(2a^2 - bc) \leq a^2(b+c)^2 \leq \left(\frac{a + (b+c)}{2}\right)^4 = 1.$$

Equality holds iff $a = b = 1, c = 0$ and permutations.

1312. **If $a, b, c > 0$ and $a + b + c = 3$, then prove that :**

$$\sum_{cyc} \frac{\left(b^5 + 2b^2c^2(b+c) + c^5\right)^5}{(b^4 + 2bc(b^2+c^2) + c^4)^3} \geq 108$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$b^4 + c^4 \overset{?}{\leq} 2\left(b^2 - bc + c^2\right)^2$$
$$\Leftrightarrow \left(b^2 + c^2\right)^2 - 2b^2c^2 \overset{?}{\leq} 2\left(b^2 + c^2\right)^2 - 4bc\left(b^2 + c^2\right) + 2b^2c^2$$
$$\Leftrightarrow \left(b^2 + c^2\right)^2 - 4bc\left(b^2 + c^2\right) + 4b^2c^2 \overset{?}{\geq} 0 \Leftrightarrow \left(b^2 + c^2 - 2bc\right)^2 \overset{?}{\geq} 0$$
$$\Leftrightarrow (b-c)^4 \overset{?}{\geq} 0 \to \text{true} \Rightarrow b^4 + 2bc\left(b^2 + c^2\right) + c^4$$
$$\leq 2\left(\left(b^2 - bc + c^2\right)^2 + bc\left(b^2 + c^2\right)\right) = 2\left(\left(b^2 + c^2\right)^2 - bc\left(b^2 + c^2\right) + b^2c^2\right)$$

$$\Rightarrow\frac{\left(b^5+2b^2c^2(b+c)+c^5\right)^5}{(b^4+2bc(b^2+c^2)+c^4)^3}\geq\frac{(b+c)^5\left(b^4-b^3c+b^2c^2-bc^3+c^4+2b^2c^2\right)^5}{8((b^2+c^2)^2-bc(b^2+c^2)+b^2c^2)^3}$$

$$=\frac{(b+c)^5\left(\left(b^2+c^2\right)^2-bc\left(b^2+c^2\right)+b^2c^2\right)^5}{8((b^2+c^2)^2-bc(b^2+c^2)+b^2c^2)^3}$$

$$=\frac{(b+c)^5}{8}.\left(\left(b^2+c^2\right)^2-bc\left(b^2+c^2\right)+b^2c^2\right)^2$$

$$=\frac{(b+c)^5}{8}.\left(x^2-\frac{xy}{2}+\frac{y^2}{4}\right)^2\ \left(x=b^2+c^2,y=2bc\right)=\frac{(b+c)^5}{8.16}.\left(3x^2+(x-y)^2\right)^2$$

$$\geq\frac{9(b+c)^5\left(b^2+c^2\right)^4}{2^7}\overset{CBS}{\geq}\frac{9(b+c)^5\left(\frac{(b+c)^2}{2}\right)^4}{2^7}$$

$$\therefore\frac{\left(b^5+2b^2c^2(b+c)+c^5\right)^5}{(b^4+2bc(b^2+c^2)+c^4)^3}\geq\frac{3^2}{2^{11}}.(b+c)^{13}\ \textit{and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{\left(b^5+2b^2c^2(b+c)+c^5\right)^5}{(b^4+2bc(b^2+c^2)+c^4)^3}\geq\frac{3^2}{2^{11}}.\sum_{cyc}(b+c)^{13}\overset{Holder}{\geq}\frac{3^2}{2^{11}.3^{12}}.\left(2\sum_{cyc}a\right)^{13}$$

$$\overset{a+b+c=3}{=}\frac{3^2.2^{13}.3^{13}}{2^{11}.3^{12}}=108\therefore\sum_{cyc}\frac{\left(b^5+2b^2c^2(b+c)+c^5\right)^5}{(b^4+2bc(b^2+c^2)+c^4)^3}\geq108$$

$\forall a,b,c>0\,\middle|\,a+b+c=3,''='' \text{ occurs iff } a=b=c=1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\sum_{cyc}\frac{\left(b^5+2b^2c^2(b+c)+c^5\right)^5}{(b^4+2bc(b^2+c^2)+c^4)^3}\geq\frac{\left[\sum_{cyc}\left(b^5+2b^2c^2(b+c)+c^5\right)\right]^5}{3\left[\sum_{cyc}\left(b^4+2bc(b^2+c^2)+c^4\right)\right]^3}$$

$$=\frac{\left[2\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a^2\right)\right]^5}{3\left[2\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)\right]^3}$$

$$=\frac{4\left(\sum_{cyc}a^3\right)^2\left(\sum_{cyc}a^2\right)^5}{3.3^3}\geq\frac{4.3^2.3^5}{81}=108,$$

the last line is true because

$$\sum_{cyc}a^3\geq\frac{(a+b+c)^3}{3^2}=3\text{ and }\sum_{cyc}a^2\geq\frac{(a+b+c)^2}{3}=3.$$

The proof is complete. Equality holds iff $a=b=c=1$.

Solution 3 by Nguyen Van Canh-BenTre-Vietnam

Firstly, we prove that

$$\boxed{a^5+2a^2b^2(a+b)+b^5 \ge \frac{(a+b)}{2}(a^4+2ab(a^2+b^2)+b^4);}$$

$$\Leftrightarrow t^5+2t^2(t+1)+1 \ge \frac{(t+1)}{2}(t^4+2t(t^2+1)+1); \left(\because t=\frac{a}{b}>0\right)$$

$$\Leftrightarrow 2\left(t^5+2t^2(t+1)+1\right)-(t+1)(t^4+2t(t^2+1)+1) \ge 0;$$

$$\Leftrightarrow t^5-3t^4+2t^3+2t^2-3t+1 \ge 0;$$

$$\Leftrightarrow (t+1)(t-1)^4 \ge 0 \text{ (true)}.$$

Secondly, we prove that

$$\boxed{a^5+2a^2b^2(a+b)+b^5 \ge \frac{6}{2^5}(a+b)^5;}$$

$$\Leftrightarrow 16\left(t^5+2t^2(t+1)+1\right) \ge 3(t+1)^5; \left(\because t=\frac{a}{b}>0\right)$$

$$\Leftrightarrow 13t^5-15t^4+2t^3+2t^2-15t+13 \ge 0;$$

$$\Leftrightarrow (t+1)(t-1)^2(13t^2-2t+13) \ge 0 \text{ (true)}.$$

$$\Rightarrow \frac{\left(a^5+2a^2b^2(a+b)+b^5\right)^5}{(a^4+2ab(a^2+b^2)+b^4)^3}$$

$$=\left(\frac{a^5+2a^2b^2(a+b)+b^5}{a^4+2ab(a^2+b^2)+b^4}\right)^3.\left(a^5+2a^2b^2(a+b)+b^5\right)^2$$

$$\ge \frac{36}{2^{13}}(a+b)^{13}; \ \forall a,b>0$$

Similary, we have

$$\frac{\left(b^5+2b^2c^2(b+c)+c^5\right)^5}{(b^4+2bc(b^2+c^2)+c^4)^3} \ge \frac{36}{2^{13}}(b+c)^{13};$$

$$\frac{\left(c^5+2c^2a^2(c+a)+a^5\right)^5}{(c^4+2ca(c^2+a^2)+a^4)^3} \ge \frac{36}{2^{13}}(c+a)^{13};$$

$$\Rightarrow \frac{\left(a^5+2a^2b^2(a+b)+b^5\right)^5}{(a^4+2ab(a^2+b^2)+b^4)^3}+\frac{\left(b^5+2b^2c^2(b+c)+c^5\right)^5}{(b^4+2bc(b^2+c^2)+c^4)^3}$$

$$+\frac{\left(c^5+2c^2a^2(c+a)+a^5\right)^5}{(c^4+2ca(c^2+a^2)+a^4)^3}$$

$$\ge \frac{36}{2^{13}}[(a+b)^{13}+(b+c)^{13}+(c+a)^{13}] \overset{\text{Holder}}{\ge} \frac{36}{2^{13}}.\frac{(2a+2b+2c)^{13}}{3^{13-1}}$$

$$=\frac{36}{2^{13}}.\frac{2^{13}3^{13}}{3^{12}}=108.$$

Proved. Equality if and only if $a=b=c=1$.

1313. **If $a,b,c>0$ such that : $a+b+c=1$, then :**

$$\sum_{cyc}\frac{a(b+c)^2}{a+1} \le \frac{1}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a(b+c)^2}{a+1} \overset{?}{\leq} \frac{5-3a}{36} \overset{b+c=1-a}{\Leftrightarrow} \frac{a(1-a)^2}{a+1} - \frac{5-3a}{36} \overset{?}{\leq} 0$$
$$\Leftrightarrow \frac{36a(1-a)^2-(5-3a)(a+1)}{36(a+1)} \overset{?}{\leq} 0 \Leftrightarrow 36a^3-69a^2+34a-5 \overset{?}{\leq} 0$$
$$\Leftrightarrow (3a-1)^2(4a-5) \overset{?}{\leq} 0 \to \text{true} \because a+b+c=1 \Rightarrow a<1 \Rightarrow 4a<4$$
$$\Rightarrow 4a-5<-1<0 \therefore \frac{a(b+c)^2}{a+1} \leq \frac{5-3a}{36} \text{ and analogs}$$
$$\Rightarrow \sum_{cyc} \frac{a(b+c)^2}{a+1} \leq \frac{3.5}{36} - \frac{a+b+c}{12} \overset{a+b+c=1}{=} \frac{5}{12} - \frac{1}{12} = \frac{1}{3}$$
$$\therefore \sum_{cyc} \frac{a(b+c)^2}{a+1} \leq \frac{1}{3}\ \forall a,b,c>0 \,\Big|\, a+b+c=1,\text{''}=\text{''} \text{ iff } a=b=c=\frac{1}{3} \text{ (QED)}$$

1314. **If $x,y,z>0$ and $0 \leq \lambda \leq 3$ then**

$$\sum_{cyc} \frac{x^3+2}{2+\lambda x+\lambda y+z^3} \geq \frac{9}{2\lambda+3}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } a := \frac{x^3+2}{3}, b := \frac{y^3+2}{3}, c := \frac{z^3+2}{3}.$$

By AM − GM inequality, we have $a \geq x,\ \ b \geq y,\ \ c \geq z$.

Then by using CBS inequality, we have

$$\sum_{cyc} \frac{x^3+2}{2+\lambda x+\lambda y+z^3} \geq \sum_{cyc} \frac{3a}{\lambda a+\lambda b+3c} \geq \frac{3(a+b+c)^2}{\sum_{cyc} a(\lambda a+\lambda b+3c)}$$
$$= \frac{9(a+b+c)^2}{3\lambda(a+b+c)^2+(3-\lambda).3(ab+bc+ca)} \geq$$
$$\geq \frac{9(a+b+c)^2}{3\lambda(a+b+c)^2+(3-\lambda)(a+b+c)^2} = \frac{9}{2\lambda+3},$$

as desired. Equality holds if and only if $a=b=c=x=y=z$ or $x=y=z=1$.

1315. **If $a,b,c>0, a+b+c=6$ then:**

$$2\sqrt{3}\sum_{cyc} a^3 + 12\sqrt{\sum_{cyc} a^2} \geq 9\sqrt{3}abc$$

Proposed by Lazaros Zachariadis-Greece

Solution 1 by Dimitris Kastriotis-Greece

$$\sum a^2 \overset{C-S}{\geq} \frac{1}{3}\left(\sum a\right)^2 = 12, \text{ Equality } a = b = c = 2$$

$$\sum a^3 \overset{Jensen}{\geq} 3\left(\frac{a+b+c}{3}\right)^3 = 24, \text{ Equality } a = b = c = 2$$

$$f(a,b,c) = 2\sqrt{3}\sum a^3 + 12\sqrt{\sum a^2} \geq 2\sqrt{3}\cdot 24 + 12\sqrt{12} = 72\sqrt{3} \quad (1)$$

$$abc \overset{AM-GM}{\leq} \left(\frac{a+b+c}{3}\right)^3 = 8, \text{ Equality } a = b = c = 2$$

$$g(a,b,c) = 9\sqrt{3}abc \leq 72\sqrt{3} \quad (2)$$

$$\begin{Bmatrix}(1)\\(2)\end{Bmatrix} \to f(a,b,c) \geq g(a,b,c) \to 2\sqrt{3}\sum a^3 + 12\sqrt{\sum a^2} \geq 9\sqrt{3}abc$$

Equality $a = b = c = 2$

Solution 2 by Khanh Hai-Vietnam

$$2\sqrt{3}\sum a^3 \geq 6\sqrt{3}abc$$

$$12\sqrt{\sum a^2} \geq 12\sqrt{\frac{\left(\sum a\right)^2}{3}} = 24\sqrt{3}$$

$$\Rightarrow LHS \geq 6\sqrt{3}abc + 24\sqrt{3} \quad (*)$$

$$RHS = 6\sqrt{3}abc + 3\sqrt{3}abc \leq 6\sqrt{3}abc + 3\sqrt{3}\cdot\frac{\left(\sum a\right)^3}{27}$$

$$= 6\sqrt{3}abc + 24\sqrt{3} \quad (**)$$

From (*) and (**) $LHS \geq 6\sqrt{3}abc + 24\sqrt{3} \geq RHS$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$2\sqrt{3}(a^3+b^3+c^3) + 12\sqrt{a^2+b^2+c^2} \geq 9\sqrt{3}abc$$

Iff $2\sqrt{3}\cdot 24\sqrt{\frac{a^2+b^2+c^2}{12}} + 12\sqrt{a^2+b^2+c^2} \geq 9\sqrt{3}abc$

Iff $24\sqrt{a^2+b^2+c^2} + 12\sqrt{a^2+b^2+c^2} \geq 9\sqrt{3}abc$

Iff $36\sqrt{a^2+b^2+c^2} \geq 9\sqrt{3}abc$

Iff $4\sqrt{a^2+b^2+c^2} \geq 3\sqrt{3}abc$

Iff $16(a^2+b^2+c^2) \geq 3(abc)^2$

Iff $16(a^2+b^2+c^2)\geq 3\times 64$

Iff $a^2+b^2+c^2\geq 12$

Because $6=a+b+c\geq 3\sqrt[3]{abc}\Leftrightarrow 6^3\geq 3^3abc\Leftrightarrow 8\geq abc$

and $6=a+b=c\Leftrightarrow (a+b+c)^2=36\Leftrightarrow a^2+b^2+c^2\geq \frac{36}{3}=12$

Therefore it is to be true

1316. Let $a_1,a_2,\dots,a_n$ be positive real numbers such that

$$a_1+a_2+\cdots+a_n=n;(\lambda\geq 0,n\in\mathbb{N}).\text{ Prove that:}$$

$$\frac{a_1}{(a_2)^2\cdot(a_1+\lambda)}+\frac{a_2}{(a_3)^2\cdot(a_2+\lambda)}+\cdots+\frac{a_n}{(a_1)^2\cdot(a_n+\lambda)}\geq\frac{n}{1+\lambda}$$

Proposed by Sidi Abdellah Lemrabott-Mauritania

Solution 1 by Tapas Das-India

$$a_1+a_2+\cdots+a_n=n$$
$$\frac{a_1+a_2+\cdots+a_n}{n}=1$$

$$AM\geq GM\ \ (a_1a_2a_3\dots a_n)^{\frac{1}{n}}\leq 1$$

$$\therefore a_1a_2\dots a_n\leq 1\quad (1)$$

$$\frac{a_1}{(a_2)^2(a_1+\lambda)}+\frac{a_2}{(a_3)^2(a_2+\lambda)}+\cdots+\frac{a_n}{(a_1)^2(a_n+\lambda)}$$

$$\overset{AM-GM}{\geq} n\left[\frac{(a_1a_2\dots a_n)}{(a_1a_2\dots a_n)^2\prod(a_1+\lambda)}\right]^{\frac{1}{n}}\geq n\left[\frac{1}{\prod(a_1+\lambda)}\right]^{\frac{1}{n}}\quad \text{(using (1))}$$

$$\overset{AM-GM}{\geq} n\left[\frac{1}{\left[\frac{\sum(a_1+\lambda)}{n}\right]^n}\right]^{\frac{1}{n}}=n\cdot\frac{n}{\sum(a_1+\lambda)}=\frac{n^2}{\sum a_1+n\lambda}=\frac{n^2}{n+n\lambda}=\frac{n}{\lambda+1}$$

$$\left(\because \sum a_1=n\right)$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum_{i=1}^{n}a_i=n$$

$$\frac{a_1}{a_2^2(a_1+\lambda)}+\frac{a_2}{a_3^2(a_2+\lambda)}+\cdots+\frac{a_n}{a_1^2(a_n+\lambda)}\ge\frac{n}{1+\lambda}$$

$$\sum_{k=1}^{n}\frac{1}{a_k}\overset{AM-GM}{\ge}n\cdot\frac{1}{\sqrt[n]{\prod a_k}}\overset{AM-GM}{\ge}\frac{n^2}{\sum a_k}=n$$

$$\sum\frac{1}{a_k}\ge n \quad (*)$$

$$I_n=\frac{\left(\frac{1}{a_2}\right)^2}{\frac{a_1+\lambda}{a_1}}+\frac{\left(\frac{1}{a_3}\right)^2}{\frac{a_2+\lambda}{a_2}}+\cdots+\frac{\left(\frac{1}{a_1}\right)^2}{\frac{a_n+\lambda}{a_n}}\overset{CBS}{\ge}\frac{\left(\sum\frac{1}{a_k}\right)^2}{n+\lambda\cdot\sum\frac{1}{a_k}}\overset{(*)}{\ge}\frac{\left(\sum\frac{1}{a_k}\right)^2}{\sum\frac{1}{a_k}\cdot(1+\lambda)}=$$

$$=\frac{\sum\frac{1}{a_k}}{1+\lambda}\overset{(*)}{\ge}\frac{n}{1+\lambda}$$

$$a_1=a_2=\cdots=a_n=1$$

Solution 3 by Remus Florin Stanca-Romania

Let $a_{n+1}=a_1$

$$\sum_{k=1}^{n}\frac{a_k}{a_{k+1}^2(a_k+\lambda)}=\sum_{k=1}^{n}\frac{\left(\frac{1}{a_{k+1}}\right)^2}{1+\frac{\lambda}{a_k}}\ge\frac{\left(\sum_{k=1}^{n}\frac{1}{a_{k+1}}\right)^2}{n+\lambda\sum_{k=1}^{n}\frac{1}{a_k}}=\frac{\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)^2}{n+\lambda\sum_{k=1}^{n}\frac{1}{a_k}}\quad (1)$$

Let $f:\mathbb{R}^+\to\mathbb{R}, f(x)=\frac{x^2}{n+\lambda x}\Rightarrow\frac{\partial f}{\partial x}=\frac{2xn+x^2\lambda}{(n+\lambda x)^2}\ge 0\Rightarrow f$ is increasing

$$\sum_{k=1}^{n}\frac{1}{a_k}\ge\frac{n^2}{\sum_{k=1}^{n}a_k}=n\Rightarrow f\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)\ge f(n)\Rightarrow\frac{\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)^2}{n+\lambda\sum_{k=1}^{n}\frac{1}{a_k}}\ge\frac{n^2}{n+\lambda n}=\frac{n}{\lambda+1}\quad (2)$$

$$\overset{(1);(2)}{\Rightarrow}\sum_{k=1}^{n}\frac{a_k}{a_{k+1}^2(a_k+\lambda)}\ge\frac{n}{\lambda+1}$$

1317. If $a,b,c>0$ then

$$\frac{a^4(a^2+bc)}{b^5+c^5}+\frac{b^4(b^2+ca)}{c^5+a^5}+\frac{c^4(c^2+ab)}{a^5+b^5}\ge a+b+c$$

Proposed by Zaza Mzhavanaze-Georgia

Solutions 1,2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 :

The given inequality can be rewritten as follows

$$\sum_{cyc}\left(\frac{a^4(a-b)(a-c)}{b^5+c^5}+\frac{a^5(b+c)}{b^5+c^5}\right)\geq a+b+c.$$

Notice that if $a \geq b \geq c$ then

$$\frac{a^4}{b^5+c^5}\geq\frac{b^4}{c^5+a^5}\geq\frac{c^4}{a^5+b^5},$$

so by Generalized Schur inequality, we deduce that

$$\sum_{cyc}\frac{a^4(a-b)(a-c)}{b^5+c^5}\geq 0.$$

So it suffices to prove that

$$\sum_{cyc}\frac{a^5(b+c)}{b^5+c^5}\geq a+b+c \ \Leftrightarrow\ \sum_{cyc}\left(\frac{a^5(b+c)}{b^5+c^5}-a\right)\geq 0$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{ab(a^4-b^4)}{b^5+c^5}-\frac{ca(c^4-a^4)}{b^5+c^5}\right)\geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{ab(a^4-b^4)}{b^5+c^5}-\frac{ab(a^4-b^4)}{c^5+a^5}\right)\geq 0$$

$$\Leftrightarrow \sum_{cyc}\frac{ab(a^4-b^4)(a^5-b^5)}{(b^5+c^5)(c^5+a^5)}\geq 0,$$

which is true because a^4-b^4 and a^5-b^5 have the same sign.
So the proof is completed. Equality holds if and only if $a=b=c$.

<u>Solution 2 :</u>
The given inequality is equivalent to

$$\sum_{cyc}(a^6+a^4bc)(a^5+b^5)(c^5+a^5)\geq(a+b+c)(a^5+b^5)(b^5+c^5)(c^5+a^5),$$

which, after expanding and simplifying becomes,

$$\sum_{cyc}a^{16}+abc\sum_{cyc}a^{13}+abc\sum_{cyc}a^5(b^8+c^8)+\sum_{cyc}a^4b^6c^6$$
$$\geq\sum_{cyc}a^{10}(b^6+c^6)+abc\sum_{cyc}a^9(b^4+c^4)+\sum_{cyc}a^6b^5c^5.$$

$$\Leftrightarrow \sum_{cyc}[a^{16}+a^4b^6c^6-a^{10}(b^6+c^6)]+abc\sum_{cyc}[a^{13}+a^5b^4c^4-a^9(b^4+c^4)]$$
$$+abc\sum_{cyc}[a^5(b^8+c^8)-2a^5b^4c^4]\geq 0$$

$$\Leftrightarrow \sum_{cyc}a^4(a^6-b^6)(a^6-c^6)+abc\sum_{cyc}a^5(a^4-b^4)(a^4-c^4)+abc\sum_{cyc}a^5(b^4-c^4)^2\geq 0$$

which is true by Generalized Schur inequality.
Equality holds if and only if $a=b=c$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$

$$\frac{a^4(a^2+bc)}{b^5+c^5}+\frac{b^4(b^2+ca)}{c^5+a^5}+\frac{c^4(c^2+ab)}{a^5+b^5}$$

$$=\frac{a^5}{b^5+c^5}\left(a+\frac{bc}{a}\right)+\frac{b^5}{c^5+a^5}\left(b+\frac{ca}{b}\right)+\frac{c^5}{a^5+b^5}\left(c+\frac{ab}{c}\right)$$

$$\geq\frac{1}{3}\left(\frac{a^5}{b^5+c^5}+\frac{b^5}{c^5+a^5}+\frac{c^5}{a^5+b^5}\right)\left(a+b+c+\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)\geq(a+b+c)$$

Iff $\frac{1}{3}\left(\frac{3}{2}\right)\left(a+b+c+\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)\geq(a+b+c)$

Iff $a+b+c+\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq 2(a+b+c)$

Iff $\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq(a+b+c)$

Iff $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$ ok

Therefore it is to be true.

1318. If $a,b,c>0, a^2+b^2+c^2=3$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Elsen Kerimov-Azerbaijan

$$a=b=c=1; 3=3$$

$$a^2+1\geq 2a\rightarrow\frac{a^2+1}{2}\geq a\rightarrow -a\geq\frac{-(a^2+1)}{2}-2-a\geq\frac{3-a^2}{2}\rightarrow\frac{1}{2-a}\leq\frac{2}{3-a^2}=$$

$$=\frac{2}{b^2+c^2}\leq\frac{2}{2bc}=\frac{1}{bc}$$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$ prove 1

$$abc\leq 1\ \ (a^2+b^2+c^2=3\quad AM\geq GM)$$

$$1\geq abc$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$$

$$1\geq abc$$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq a+b+c$ prove 2 $\rightarrow\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c)\geq 9\rightarrow$

$\rightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{9}{a+b+c}\geq a+b+c$ prove 3 $\rightarrow 3\geq a+b+c$ prove 4

$$a^2+1\geq 2a,\qquad b^2+1\geq 2b,\qquad c^2+1\geq 2c$$
$$3\geq a+b+c$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c,\ \ q:=ab+bc+ca,\ \ r:=abc$.

From the given condition, we have $q=\dfrac{p^2-3}{2}$.

Since $a^2+b^2+c^2<(a+b+c)^2\leq 3(a^2+b^2+c^2)$, then $\sqrt{3}\leq \mathrm{p}\leq 3$,

and since $(ab+bc+ca)^2\geq 3abc(a+b+c)$, then, $r\leq \dfrac{q^2}{3p}=\dfrac{(p^2-3)^2}{12p}$.

Using these results, we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=\frac{q}{r}=\frac{p^2-3}{2r}\geq \frac{6p}{p^2-3},$$

and,

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}=\frac{12-4(a+b+c)+(ab+bc+ca)}{8-4(a+b+c)+2(ab+bc+ca)-abc}=\frac{12-4p+q}{8-4p+2q-r}$$
$$\leq \frac{12-4p+\frac{p^2-3}{2}}{8-4p+p^2-3-\frac{(p^2-3)^2}{12p}}=\frac{6p(21-8p+p^2)}{-9+60p-42p^2+12p^3-p^4}.$$

So it suffices to prove that

$$\frac{6p}{p^2-3}\geq \frac{6p(21-8p+p^2)}{-9+60p-42p^2+12p^3-p^4}$$
$$\Leftrightarrow\ -9+60p-42p^2+12p^3-p^4-(p^2-3)(21-8p+p^2)\geq 0$$
$$\Leftrightarrow 54+36p-60p^2+20p^3-2p^4\geq 0$$
$$\Leftrightarrow 2(3-p)^2[3+p+p(3-p)]\geq 0,\ \ \textit{which is true}.$$

Equality holds iff $a=b=c=1$.

1319. If $a,\mathrm{b},\mathrm{c}>0$, *then prove that*:

$$4\left(\frac{\sqrt{a}}{\mathrm{b}+\mathrm{c}}+\frac{\sqrt{\mathrm{b}}}{\mathrm{c}+a}+\frac{\sqrt{\mathrm{c}}}{a+\mathrm{b}}\right)^6(a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+a)+52\left(\frac{a}{\mathrm{b}+\mathrm{c}}+\frac{\mathrm{b}}{\mathrm{c}+a}+\frac{\mathrm{c}}{a+\mathrm{b}}\right)^3\geq 1215$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $\mathrm{b}+\mathrm{c}=x,\mathrm{c}+a=\mathrm{y},a+\mathrm{b}=\mathrm{z}\Rightarrow x+\mathrm{y}-\mathrm{z}=2\mathrm{c}>0$,
$\mathrm{y}+\mathrm{z}-x=2a>0$ and $\mathrm{z}+x-\mathrm{y}=2\mathrm{b}>0\Rightarrow x+\mathrm{y}>z,\mathrm{y}+\mathrm{z}>x,\mathrm{z}+x>y$
$\Rightarrow x,\mathrm{y},\mathrm{z}$ form sides of a triangle with semiperimeter, circumradius and inradius
$=\mathrm{s},\mathrm{R},\mathrm{r}$ (say) yielding $2\sum_{\mathrm{cyc}}a=\sum_{\mathrm{cyc}}x=2\mathrm{s}\Rightarrow\sum_{\mathrm{cyc}}a=\mathrm{s}$
$\Rightarrow a=\mathrm{s}-x,\mathrm{b}=\mathrm{s}-\mathrm{y},\mathrm{c}=\mathrm{s}-\mathrm{z}$

$$\therefore \frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b}=\sum_{cyc}\frac{\sqrt{s-x}}{x}\overset{A-G}{\geq} 3\sqrt[3]{\frac{\sqrt{(s-x)(s-y)(s-z)}}{xyz}}=3\sqrt[3]{\frac{r\sqrt{s}}{4Rrs}}$$

$$=3\sqrt[3]{\frac{1}{4R\sqrt{s}}}\Rightarrow 4\left(\frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b}\right)^6(a+b)(b+c)(c+a)\geq\frac{4.729.4Rrs}{16R^2s}$$

$$\therefore 4\left(\frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b}\right)^6(a+b)(b+c)(c+a)\overset{(\bullet)}{\geq}\frac{729r}{R}$$

$$\text{Again, } 252\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^3=252\left(\sum_{cyc}\frac{s-x}{x}\right)^3=252\left(\frac{s\sum_{cyc}yz}{xyz}-3\right)^3$$

$$=252\left(\frac{s(s^2+4Rr+r^2)}{4Rrs}-3\right)^3=252\left(\frac{s^2-8Rr+r^2}{4Rr}\right)^3\overset{Gerretsen}{\geq}$$

$$252\left(\frac{8Rr-4r^2}{4Rr}\right)^3\therefore 252\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^3\overset{(\bullet\bullet)}{\geq}\frac{252(2R-r)^3}{R^3}$$

$$\therefore (\bullet)+(\bullet\bullet)\Rightarrow LHS\geq\frac{252(2R-r)^3+729R^2r}{R^3}\overset{?}{\geq}1215$$

$$\Leftrightarrow 89t^3-255t^2+168t-28\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big(50t^2+39t(t-2)+t+14\big)\overset{?}{\geq}0\rightarrow \text{true} \because t\overset{Euler}{\geq}2$$

$$\Rightarrow 4\left(\frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b}\right)^6(a+b)(b+c)(c+a)$$

$$+252\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^3\geq 1215, ''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$4\left(\frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b}\right)^6(a+b)(b+c)(c+a)+252\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^3\geq 1215\quad (*)$$

$$\text{Let } x:=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

$$\text{By Nesbitt's inequality, we have } x\geq\frac{3}{2}.$$

Also, by Schur's inequality, we have

$$x=\frac{(a^3+b^3+c^3+3abc)+ab(a+b)+bc(b+c)+ca(c+a)}{(a+b)(b+c)(c+a)}$$

$$\geq\frac{2[ab(a+b)+bc(b+c)+ca(c+a)]}{(a+b)(b+c)(c+a)}=2-\frac{4abc}{(a+b)(b+c)(c+a)},$$

$$\text{then } \frac{4abc}{(a+b)(b+c)(c+a)}\geq 2-x.$$

Now, by AM − GM inequality, we have

$$\left(\frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b}\right)^6\geq\left(3\sqrt[3]{\frac{\sqrt{a}}{b+c}\cdot\frac{\sqrt{b}}{c+a}\cdot\frac{\sqrt{c}}{a+b}}\right)^6=\frac{3^6.abc}{[(a+b)(b+c)(c+a)]^2}$$

$$\geq\frac{3^6(2-x)}{4(a+b)(b+c)(c+a)}.$$

Using these results, we obtain

$$LHS_{(*)}\geq 3^6(2-x)+252x^3=1215+9(2x-3)(14x^2+21x-9)\geq 1215,$$

because $x\geq\frac{3}{2}$. **So the proof is complete. Equality holds iff** $a=b=c$.

1320. **If $a,b,c>0, a+b+c=3$, then :**

$$\sum_{cyc}\frac{1}{a^3+b^3}+3\sum_{cyc}\frac{1}{ab(a+b)}\geq 6$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0,$

$y+z-x=2a>0$ **and** $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$

$\Rightarrow x,y,z$ **form sides of a triangle with semiperimeter, circumradius and inradius**

$$=s,R,r\ (say)\text{ yielding }2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)$$

$$\Rightarrow a=s-x, b=s-y, c=s-z\text{ and }\sum_{cyc}a^3=$$

$$\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)\overset{via\ (1)}{=}s^3-3xyz\Rightarrow\sum_{cyc}a^3=s^3-12Rrs\rightarrow(2)$$

$$3\sum_{cyc}\frac{1}{ab(a+b)}\overset{3=a+b+c}{=}\sum_{cyc}\frac{a+b+c}{ab(a+b)}=\sum_{cyc}\frac{1}{ab}+\frac{1}{abc}\sum_{cyc}\frac{a^2}{b+c}$$

$$=\frac{\sum_{cyc}a}{abc}+\frac{1}{r^2s}\sum_{cyc}\frac{(s-x)^2}{x}\overset{a+b+c=3}{=}\frac{3}{r^2s}+\frac{1}{r^2s}\sum_{cyc}\frac{s^2-2sx+x^2}{x}$$

$$=\frac{3}{r^2s}+\frac{1}{r^2s}\left(\frac{s^2}{xyz}\sum_{cyc}xy-2s\sum_{cyc}1+\sum_{cyc}x\right)$$

$$=\frac{3}{r^2s}+\frac{1}{r^2s}\left(\frac{s}{4Rr}(s^2+4Rr+r^2)-6s+2s\right)=\frac{3}{r^2s}+\frac{1}{r^2}\cdot\frac{s^2-12Rr+r^2}{4Rr}$$

$$\overset{a+b+c=3}{=}\frac{3}{r^2s}+\frac{3}{r^2}\cdot\frac{s^2-12Rr+r^2}{4Rr(\sum_{cyc}a)}\overset{via\ (1)}{=}\frac{3}{r^2s}+\frac{3(s^2-12Rr+r^2)}{4Rr^3s}$$

$$\Rightarrow\sum_{cyc}\frac{1}{a^3+b^3}+3\sum_{cyc}\frac{1}{ab(a+b)}\overset{Bergstrom}{\geq}\frac{9}{2\sum_{cyc}a^3}+\frac{3}{r^2s}+\frac{3(s^2-12Rr+r^2)}{4Rr^3s}$$

$$\overset{\text{via }(2)}{=}\frac{9}{2s(s^2-12Rr)}+\frac{3}{r^2s}+\frac{3(s^2-12Rr+r^2)}{4Rr^3s}\overset{?}{\geq}6\overset{3=a+b+c}{=}\frac{6.27}{\left(\sum_{cyc}a\right)^3}\overset{\text{via }(1)}{=}\frac{162}{s^3}$$

$$\Leftrightarrow\frac{6Rr^3+4Rr(s^2-12Rr)+(s^2-12Rr)(s^2-12Rr+r^2)}{4Rr^3s(s^2-12Rr)}\overset{?}{\geq}\frac{54}{s^3}\overset{\text{expanding and re-arranging}}{\Leftrightarrow}$$

$$\boxed{s^6-(20Rr-r^2)s^4+r^2s^2(96R^2-222Rr)+2592R^2r^4\underset{(*)}{\overset{?}{\geq}}0}\text{ and}$$

$$\because(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\ \therefore\text{ in order to prove }(*),\text{it suffices to prove :}$$

$$\text{LHS of }(*)\geq(s^2-16Rr+5r^2)^3\overset{\text{expanding and re-arranging}}{\Leftrightarrow}$$

$$\boxed{\begin{array}{c}(28Rr-14r^2)s^4-r^2s^2(672R^2-258Rr+75r^2)+\\ r^3(4096R^3-1248R^2r+1200Rr^2-125r^3)\underset{(**)}{\overset{?}{\geq}}0\end{array}}\text{ and}$$

$$\because(28Rr-14r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore\text{ in order to prove }(**),\text{it suffices}$$

$$\text{to prove : LHS of }(**)\geq(28Rr-14r^2)(s^2-16Rr+5r^2)^2$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow}$$

$$\boxed{(224R^2-470Rr+65r^2)s^2\overset{(***)}{\geq}r(3072R^3-6816R^2r+1740Rr^2-225r^3)}\text{ and}$$

$$224R^2-470Rr+65r^2=(R-2r)(224R-22r)+21r^2\overset{\text{Euler}}{\geq}21r^2>0$$

$$\therefore\text{LHS of }(***)\overset{\text{Gerretsen}}{\geq}(224R^2-470Rr+65r^2)(16Rr-5r^2)\overset{?}{\geq}$$

$$r(3072R^3-6816R^2r+1740Rr^2-225r^3)\overset{\text{expanding and simplifying}}{\Leftrightarrow}$$

$$256t^3-912t^2+825t-50\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(56t^2+200t(t-2)+25)\overset{?}{\geq}0$$

$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}\Rightarrow\sum_{cyc}\frac{1}{a^3+b^3}+3\sum_{cyc}\frac{1}{ab(a+b)}$$

$$\geq6\ \forall\,a,b,c>0\,\Big|\sum_{cyc}a=3,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

1321.

$$a,b,c>0\Rightarrow352\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^5+405\sqrt[4]{\frac{27abc}{(a+b+c)^3}}\geq3078$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=\dfrac{(a+b+c)^2}{3(ab+bc+ca)}\geq1$. By the $AM-GM$ *inequality*, we have :

$$\frac{27abc}{(a+b+c)^3}\leq1,\ \text{ then, }\ \sqrt[4]{\frac{27abc}{(a+b+c)^3}}\geq\frac{27abc}{(a+b+c)^3}.$$

$$\text{Also, by Schur's inequality, we have :}$$

$$9abc \geq 4(a+b+c)(ab+bc+ca)-(a+b+c)^3, \ \ then, \ \ \frac{27abc}{(a+b+c)^3} \geq \frac{4}{x}-3.$$

$$\text{Now, by CBS inequality, we have :}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} = \frac{3x}{2}.$$

$$\text{Therefore,}$$

$$352\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^5+405.\sqrt[4]{\frac{27abc}{(a+b+c)^3}} \geq 352\left(\frac{3x}{2}\right)^5+405\left(\frac{4}{x}-3\right)$$

$$= 2349x^5+324\left(x^5+\frac{5}{x}\right)-1215$$

$$\overset{x\geq 1 \ and \ AM-GM}{\geq} \ \ 2349+324\times 6\sqrt[6]{x^5.\left(\frac{1}{x}\right)^5}-1215 = 3078.$$

$$\text{So the proof is completed. Equality holds iff } a=b=c.$$

Solution 2 by Christos Tsifakis-Greece

Lemma:

$$\forall x,y,z>0 \Rightarrow (x+y)(y+z)(z+x) \geq \sqrt[4]{\frac{4096\left(xyz(x+y+z)\right)^3}{27}}.$$

By Popoviciu for $f(x)=\ln x$ in $(0,+\infty) \Rightarrow$

$$f(x)+f(y)+f(z)+3-f\left(\frac{x+y+z}{3}\right) \leq 2\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right) \quad \textbf{(1)}$$

By Jensen $\Rightarrow$

$$\left\{f(x)+f(y) \leq 2f\left(\frac{x+y}{2}\right), f(y)+f(z) \leq 2f\left(\frac{y+z}{2}\right), f(z)+f(x) \leq 2f\left(\frac{z+x}{2}\right)\right\} \quad \textbf{(2)}$$

$$\textbf{(1)}+\textbf{(2)} \Rightarrow 3\big(f(x)+f(y)+f(z)\big)+3f\left(\frac{x+y+z}{3}\right) \leq 4\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right) \Rightarrow$$

$$3(\ln x+\ln y+\ln z)+3\ln\left(\frac{x+y+z}{3}\right) \leq 4\left(\ln\left(\frac{x+y}{2}\right)+\ln\left(\frac{y+z}{2}\right)+\ln\left(\frac{z+x}{2}\right)\right) \Rightarrow$$

$$\ln\left((xyz)^3\left(\frac{x+y+z}{3}\right)^3\right) \leq \ln\left(\frac{(x+y)(y+z)(z+x)}{8}\right)^4 \Rightarrow$$

$$(x+y)(y+z)(z+x) \ge \sqrt[4]{\frac{4096\big(xyz(x+y+z)\big)^3}{27}}$$

the equality for $x = y = z$.

$$\left.\begin{array}{r} Let\ x = \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} \overset{Nesbitt}{\ge} \dfrac{3}{2} \Rightarrow 10x \ge 15 \\ By\ Schur \Rightarrow x + \dfrac{4abc}{\prod_{cyc}(a+b)} \ge 2 \end{array}\right\} \overset{(+)}{\Rightarrow}$$

$$\Rightarrow 11x + \frac{4abc}{\prod_{cyc}(a+b)} \ge 17 \Rightarrow \frac{4abc}{\prod_{cyc}(a+b)} \ge 17 - 11x \Rightarrow$$

$$\frac{4abc}{\prod_{cyc}(a+b)} \ge 17 - 11x \Rightarrow \frac{176x^5}{405} + \frac{4abc}{\prod_{cyc}(a+b)} \ge$$

$$\ge 17 - 11x + \frac{176x^5}{405} \overset{\left(17-11x+\frac{176x^5}{405} \ge 3,8\right)}{\Rightarrow}$$

$$\frac{176x^5}{405} + \frac{4abc}{\prod_{cyc}(a+b)} \ge 3,8 \overset{Lemma}{\Rightarrow} \frac{176x^5}{405} + \frac{4abc}{\sqrt[4]{\dfrac{2^{12}\big(abc(a+b+c)\big)^3}{3^3}}} \ge 3,8 \Rightarrow$$

$$\frac{176x^5}{405} + \frac{abc}{2(abc)^{\frac{3}{4}}\sqrt[4]{\left(\dfrac{a+b+c}{3}\right)^3}} \ge 3,8 \Rightarrow$$

$$\Rightarrow \frac{176x^5}{405} + \frac{1}{2}\sqrt[4]{\frac{27abc}{(a+b+c)^3}} \ge 3,8 \Rightarrow$$

$$352\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^5 + 405 - \sqrt[4]{\frac{27abc}{(a+b+c)^3}} \ge 3078,$$

the equality for $\left(x = \frac{3}{2}\right) \wedge (a = b = c) \Leftrightarrow a = b = c$.

1322. If $a, b, c > 0, n \in \mathbb{N}, a^n + b^n + c^n = 3$ then:

$$\frac{a^n(a^2+bc)}{(b+c)^2} + \frac{b^n(b^2+ca)}{(c+a)^2} + \frac{c^n(c^2+ab)}{(a+b)^2} \ge \frac{3}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Sidi Abdellah Lemrabott-Mauritania

WLOG, we assume that: $(a \geq b \geq c)$

$$\sum_{cyc}\left(\frac{a^n(a^2+bc)}{(b+c)^2}-\frac{a^n}{2}\right)=\sum_{cyc}\left[\left(\frac{a^n}{2(b+c)^2}\right)\left((a-b)(a+b)+(a-c)(a+c)\right)\right]$$

pose: $x=\frac{a^n}{2(b+c)^2}, y=\frac{b^n}{2(c+a)^2}, z=\frac{c^n}{2(a+b)^2}$

then clearly: $(x \geq y \geq z$ because: $a \geq b \geq c)$:

$$x(a-b)(a+b)+x(a-c)(a+c)+y(b-a)(b+a)+y(b-c)(b+c)+$$

$$+z(c-a)(c+a)+z(c-b)(c+b)=$$

$$=(a+b)(a-b)(x-y)+(b+c)(b-c)(y-z)+(c+a)(a-c)(x-z)\geq 0$$

$$\Leftrightarrow \sum_{cyc}\frac{a^n(a^2+bc)}{(b+c)^2}\geq\frac{a^n+b^n+c^n}{2}=\frac{3}{2}$$

Equality holds if: $a=b=c$. (Q.E.D)

Solution 2 by Sidi Abdellah Lemrabott-Mauritania

WLOG, we assume that: $(a \geq b \geq c), n \geq 1$:

if:$n=1, a+b+c=3$ we have:

$$\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b} \text{ and } \frac{a(a^2+bc)}{b+c}\geq\frac{b(b^2+ca)}{c+a}\geq\frac{c(c^2+ab)}{a+b}:$$

$$\sum_{cyc}\left(\frac{a(a^2+bc)}{b+c}-a^2\right)=\sum_{cyc}\frac{a(a-b)(a-c)}{b+c}\geq 0$$

$$\Leftrightarrow \sum_{cyc}\frac{a(a^2+bc)}{b+c}\geq\sum_{cyc}a^2\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{3}\overset{a+b+c=3}{=}3\Leftrightarrow\sum_{cyc}\frac{a(a^2+bc)}{b+c}\geq 3$$

$$LHS=\sum_{cyc}\frac{a(a^2+bc)}{(b+c)^2}=\sum_{cyc}\left(\frac{1}{b+c}\right)\left(\frac{a(a^2+bc)}{b+c}\right)\overset{Chebyshev\ \&\ Bergstrom}{\geq}$$

$$\geq\frac{1}{3}\left(\frac{9}{2(a+b+c)}\right)\cdot(3)\overset{a+b+c=3}{=}\frac{3}{2}$$

pose: $n=k$ is true, know we need to prove: $n=k+1$ is true,

$$a^{n+1}+b^{n+1}+c^{n+1}=3$$

$$\sum_{cyc}\left(\frac{a^n(a^2+bc)}{b+c}-a^{n+1}\right)=\sum_{cyc}\frac{a^n(a-b)(a-c)}{b+c}\ge 0$$

$$\Leftrightarrow \sum_{cyc}\frac{a^n(a^2+bc)}{b+c}\overset{a^{n+1}+b^{n+1}+c^{n+1}=3}{\ge} 3 \quad (1)$$

$a\ge b\ge c$ then:

$\frac{a}{b+c}\ge\frac{b}{c+a}\ge\frac{c}{a+b}$ and $\frac{a^n(a^2+bc)}{b+c}\ge\frac{b^n(b^2+ca)}{c+a}\ge\frac{c^n(c^2+ab)}{a+b}$:

$$LHS=\sum_{cyc}\left(\frac{a}{b+c}\cdot\frac{a^n(a^2+bc)}{(b+c)}\right)\overset{Chebyshev}{\ge}$$

$$\ge\frac{1}{3}\left(\sum_{cyc}\frac{a}{b+c}\right)\left(\sum_{cyc}\frac{a^n(a^2+bc)}{b+c}\right)\overset{Nesbitt\ \&\ (1)}{\ge}\frac{1}{3}\cdot\frac{3}{2}\cdot 3=\frac{3}{2}$$

Equality holds if $(a=b=c)$. (Q.E.D.)

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{\left(\frac{a^n(a^2+bc)}{b+c}\right)^2}{a^n(a^2+bc)}+\frac{\left(\frac{b^n(b^2+ca)}{c+a}\right)^2}{b^n(b^2+ca)}+\frac{\left(\frac{c^n(c^2+ab)}{a+b}\right)^2}{c^n(c^2+ab)}\ge\frac{3}{2}$$

$$\text{Iff }\frac{\left(\frac{a^n(a^2+bc)}{(b+c)}+\frac{b^n(b^2+ca)}{(c+a)}+\frac{c^n(c^2+ab)}{(a+b)}\right)^2}{a^n(a^2+bc)+b^n(b^2+ca)+c^n(c^2+ab)}\ge\frac{3}{2}$$

$$\text{Iff }\frac{\left(\frac{\left(a^n(a^2+bc)+b^n(b^2+ca)+c^n(c^2+ab)\right)^2}{a^n(a^2+bc)(b+c)+b^n(b^2+ca)(c+a)+c^n(c^2+ab)(a+b)}\right)^2}{a^n(a^2+bc)+b^n(b^2+ca)+c^n(c^2+ab)}\ge\frac{3}{2}$$

$$\text{Iff }\left(a^n(a^2+bc)+b^n(b^2+ca)+c^n(c^2+ab)\right)^3\ge$$

$$\ge\frac{3}{2}\left(a^n(a^2+bc)(b+c)+b^n(b^2+ca)(c+a)+c^n(c^2+ab)(a+b)\right)^2$$

$$\text{Iff }\left(a^n(a^2+bc)+b^n(b^2+ca)+c^n(c^2+ab)\right)^3\ge$$

$$\ge\frac{3}{2}\left[\frac{(a^n(a^2+ba)+b^n(b^2+ca)+c^n(c^2+ab)\left(2(a+b+c)\right)}{3}\right]^2$$

$$\text{Iff }a^n(a^2+bc)+b^n(b^2+ca)+c^n(c^2+ab)\ge\frac{2}{3}(a+b+c)^2$$

Iff $\frac{(a^n+b^n+c^n)}{3}(a^2+b^2+c^2+ab+bc+ca)\geq\frac{2}{3}(a+b+c)^2$

$$a^n+b^n+c^n=3$$

Iff $3(a^2+b^2+c^2+ab+bc+ca)\geq 2(a^2+b^2+c^2)$

Iff $a^2+b^2+c^2\geq ab+bc+ca$

Therefore it is to be true

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{a^n(a^2+bc)}{(b+c)^2}=\sum_{cyc}\frac{a^n\{(a-b)(a-c)+a(b+c)\}}{(b+c)^2}$$

$$=\sum_{cyc}\frac{a^n(a-b)(a-c)}{(b+c)^2}+\sum_{cyc}\frac{a^{n+1}}{b+c}\geq\sum_{cyc}\frac{a^{n+1}}{b+c}$$

Applying Schur's Inequality

$$x(a-b)(a-c)+y(b-a)(b-c)+z(c-a)(c-b)\geq 0$$

we have

$$\sum_{cyc}\frac{a^n(a-b)(a-c)}{(b+c)^2}\geq 0$$

where

$x=\frac{a^n}{(b+c)^2}, y=\frac{b^n}{(c+a)^2}$ and $z=\frac{c^n}{(a+b)^2}$

Let $\frac{a^n}{(b+c)^2}\geq\frac{b^n}{(c+a)^2}\geq\frac{c^n}{(a+b)^2}$ then $a\geq b\geq c$

$$\overset{\substack{Chebyshev's\\ Inequality}}{\geq}\frac{1}{3}\left(\sum_{cyc}\frac{a}{b+c}\right)\left(\sum_{cyc}a^n\right)\geq\frac{1}{3}\cdot\frac{3}{2}\cdot 3=\frac{3}{2}\text{ (proved)}$$

Equality at $a=b=c=1$.

Solution 5 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum a^n=3:\frac{\sum a^n(a^2+bc)}{(b+c)^2}\geq\frac{3}{2}$$

$$a\geq b\geq c:\frac{a}{b+c}\geq\frac{b}{c+a}\geq\frac{c}{a+b}$$

$$RHS=\frac{3}{2}=\frac{3}{2}\cdot\frac{\sum a^4}{3}\leq\sum\frac{a}{b+c}\cdot\frac{\sum a^4}{3}\overset{Chebyshev}{\leq}$$

$$\leq 3\cdot\frac{1}{3}\cdot\sum a^n\left(\frac{a}{b+c}\right)=\sum\frac{a^n\big(a(b+c)\big)}{(b+c)^2}=RHS$$

$$\sum\frac{a^n(a^2+bc)}{(b+c)^2}\overset{?}{\geq}\sum\frac{a^n\big(a(b+c)\big)}{(b+c)^2}$$

$$\sum\frac{a^n}{(b+c)^2}\cdot(a^2+bc-ab-ac)\geq 0$$

$$\sum\frac{a^n}{(b+c)^2}(a-b)(a-c)\geq 0$$

$$\frac{a^4}{(b+c)^2}\geq\frac{b^n}{(c+a)^2}\geq\frac{c^n}{(a+b)^2}\qquad (*)$$

$$\frac{a^n}{(b+c)^2}(a-b)(a-c)+\frac{b^n}{(a+c)^2}(b-a)(b-c)+\frac{c^n(c-a)(c-b)}{(a+b)^2}\geq$$

$$\overset{(*)}{\geq}\frac{a^n}{(b+c)^2}(a-b)(a-c)+\frac{a^n}{(b+c)^2}(b-a)(b-c)+\frac{c^n}{(a+b)^2}(c-a)(c-b)=$$

$$=\underbrace{\frac{a^n}{(b+c)^2}\cdot(a-b)^2}_{\geq 0}+\underbrace{\frac{c^n}{(a+b)^2}(c-a)(c-b)}_{\geq 0}\geq 0$$

$$=\frac{a^n}{(b+c)^2}\cdot(a-b)^2+\frac{c^n}{(a+b)^2}\cdot(c-a)(c-b)\geq 0$$

$$(a=b=c)$$

1323. If $a,b,c>0$, *then prove that* :

$$\frac{a^8(a^2+bc)}{(b+c)^{10}}+\frac{b^8(b^2+ca)}{(c+a)^{10}}+\frac{c^8(c^2+ab)}{(a+b)^{10}}\geq\frac{3}{512}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Via Power}-\text{Mean inequality},\left(\frac{1}{3}\cdot\sum_{cyc}a^{\frac{20}{11}}\right)^{\frac{11}{20}}\geq\frac{1}{3}\cdot\sum_{cyc}a$$

$$\Rightarrow\left(\sum_{cyc}a^{\frac{20}{11}}\right)^{11}\overset{(i)}{\geq}\frac{1}{3^9}\left(\sum_{cyc}a\right)^{20}\ \text{and also},$$

$$\left(\frac{1}{3}\cdot\sum_{cyc}a^{\frac{17}{11}}\right)^{\frac{11}{17}}\geq\frac{1}{3}\cdot\sum_{cyc}a\Rightarrow\left(\sum_{cyc}a^{\frac{17}{11}}\right)^{11}\overset{(ii)}{\geq}\frac{1}{3^6}\left(\sum_{cyc}a\right)^{17}$$

$$\frac{a^8(a^2+bc)}{(b+c)^{10}}+\frac{b^8(b^2+ca)}{(c+a)^{10}}+\frac{c^8(c^2+ab)}{(a+b)^{10}}=\sum_{cyc}\frac{a^{10}}{(b+c)^{10}}+abc\sum_{cyc}\frac{a^7}{(b+c)^{10}}$$

$$=\sum_{cyc}\frac{\left(a^{\frac{20}{11}}\right)^{11}}{(ab+ac)^{10}}+abc\sum_{cyc}\frac{\left(a^{\frac{17}{11}}\right)^{11}}{(ab+ac)^{10}}\overset{\text{Radon}}{\geq}\frac{\left(\sum_{cyc}a^{\frac{20}{11}}\right)^{11}}{2^{10}\left(\sum_{cyc}ab\right)^{10}}+abc\frac{\left(\sum_{cyc}a^{\frac{17}{11}}\right)^{11}}{2^{10}\left(\sum_{cyc}ab\right)^{10}}$$

$$\overset{\text{via (i),(ii)}}{\geq}\frac{\frac{1}{3^9}\left(\sum_{cyc}a\right)^{20}}{2^{10}\left(\sum_{cyc}ab\right)^{10}}+abc.\frac{\frac{1}{3^6}\left(\sum_{cyc}a\right)^{17}}{2^{10}\left(\sum_{cyc}ab\right)^{10}}\overset{?}{\geq}\frac{3}{512}$$

$$\Leftrightarrow \boxed{\frac{\frac{1}{3^9}\left(\sum_{cyc}a\right)^{20}}{\left(\sum_{cyc}ab\right)^{10}}+abc.\frac{\frac{1}{3^6}\left(\sum_{cyc}a\right)^{17}}{\left(\sum_{cyc}ab\right)^{10}}\underset{(*)}{\overset{?}{\geq}}6}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)\Rightarrow a=s-x, b=s-y,$ $c=s-z$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\rightarrow(2)\therefore(1),(2)\text{ and }abc=\prod_{cyc}(s-x)=r^2s$$

$$\Rightarrow(*)\Leftrightarrow\boxed{\frac{s^{20}}{3^9(4Rr+r^2)^{10}}+\frac{r^2s^{18}}{3^6(4Rr+r^2)^{10}}\overset{(**)}{\geq}6}$$

$$\text{Now, }s^2\overset{\text{Gerretsen}}{\geq}3(4Rr+r^2)+4r(R-2r)\overset{\text{Euler}}{\geq}3(4Rr+r^2)$$

$$\therefore s^{14}\geq 3^7(4Rr+r^2)^7\text{ and }s^{12}\geq 3^6(4Rr+r^2)^6$$

$$\Rightarrow\text{LHS of }(**)\geq\frac{s^6.3^7(4Rr+r^2)^7}{3^9(4Rr+r^2)^{10}}+\frac{r^2s^6.3^6(4Rr+r^2)^6}{3^6(4Rr+r^2)^{10}}$$

$$=\frac{s^6}{9r^3(4R+r)^3}+\frac{s^6}{r^2(4R+r)^4}\overset{?}{\geq}6\Leftrightarrow\frac{(4R+r)s^6+9rs^6}{9r^3(4R+r)^4}\overset{?}{\geq}6$$

$$\Leftrightarrow\boxed{(2R+5r)s^6\underset{(***)}{\overset{?}{\geq}}27r^3(4R+r)^4}$$

$$\text{Again, }(2R+5r)s^6\overset{\text{Gerretsen}}{\geq}(2R+5r)(16Rr-5r^2)^3\overset{?}{\geq}27r^3(4R+r)^4$$

$$\Leftrightarrow 640t^4+2944t^3-9696t^2+2659t-326\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(640t^3+3600t^2+624t(t-2)+163\right)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\frac{a^8(a^2+bc)}{(b+c)^{10}}+\frac{b^8(b^2+ca)}{(c+a)^{10}}+\frac{c^8(c^2+ab)}{(a+b)^{10}}\geq\frac{3}{512},$$

" = " iff $a=b=c$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $\frac{a^8(a^2+bc)}{(b+c)^{10}}=\frac{a^8}{(b+c)^{10}}(a-b)(a-c)+\left(\frac{a}{b+c}\right)^9$ (and analogs)

WLOG, we may assume that $a\ge b\ge c$. We have $\frac{a^8}{(b+c)^{10}}\ge\frac{b^8}{(c+a)^{10}}\ge\frac{c^8}{(a+b)^{10}}$, then by the Generalized Schur inequality, we have

$$\sum_{cyc}\frac{a^8}{(b+c)^{10}}(a-b)(a-c)\ge 0.$$

Therefore

$$\sum_{cyc}\frac{a^8(a^2+bc)}{(b+c)^{10}}\ge\sum_{cyc}\left(\frac{a}{b+c}\right)^9\overset{Hölder}{\ge}\frac{1}{3^8}\cdot\left(\sum_{cyc}\frac{a}{b+c}\right)^9\overset{Nesbitt}{\ge}\frac{1}{3^8}\cdot\left(\frac{3}{2}\right)^9=\frac{3}{512}.$$

Equality holds if and only if $a=b=c$.

1324. If $a,b,c>0, n\in\mathbb{N}$ then

$$\frac{a^n(a^2+bc)}{b^{n+1}+c^{n+1}}+\frac{b^n(b^2+ca)}{c^{n+1}+a^{n+1}}+\frac{c^n(c^2+ab)}{a^{n+1}+b^{n+1}}\ge a+b+c$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Bui Hong Suc-Vietnam

By Rearrangement inequality:

$$LHS=\frac{a^n(a^2+bc)}{b^{n+1}+c^{n+1}}+\frac{b^n(b^2+ac)}{c^{n+1}+a^{n+1}}+\frac{c^n(c^2+ab)}{a^{n+1}+b^{n+1}}$$

$$=\frac{a^{n+1}\cdot a}{b^{n+1}+c^{n+1}}+\frac{a^n\cdot b\cdot c}{b^{n+1}+c^{n+1}}+\frac{b^{n+1}\cdot b}{c^{n+1}+a^{n+1}}+\frac{b^n\cdot a\cdot c}{c^{n+1}+a^{n+1}}+\frac{c^{n+1}\cdot c}{a^{n+1}+b^{n+1}}+$$
$$+\frac{c^n\cdot a\cdot b}{a^{n+1}+b^{n+1}}$$

$$\ge\frac{c^{n+1}\cdot a}{b^{n+1}+c^{n+1}}+\frac{a^{n+1}\cdot c}{a^{n+1}+b^{n+1}}+\frac{b^{n+1}\cdot a}{a^{n+1}+b^{n+1}}+\frac{c^n\cdot b\cdot b}{c^{n+1}+a^{n+1}}+$$
$$+\frac{b^{n+1}\cdot c}{b^{n+1}+c^{n+1}}+\frac{a^{n+1}\cdot c}{c^{n+1}+a^{n+1}}$$

$$\ge\left(\frac{c^{n+1}\cdot a}{b^{n+1}+c^{n+1}}+\frac{b^{n+1}\cdot a}{b^{n+1}+c^{n+1}}\right)+\left(\frac{a^{n+1}\cdot c}{a^{n+1}+b^{n+1}}+\frac{b^{n+1}\cdot c}{a^{n+1}+b^{n+1}}\right)+$$
$$+\left(\frac{c^{n+1}\cdot b}{c^{n+1}+a^{n+1}}+\frac{a^{n+1}\cdot b}{c^{n+1}+a^{n+1}}\right)$$

$$=a\left(\frac{c^{n+1}+b^{n+1}}{b^{n+1}+c^{n+1}}\right)+c\left(\frac{a^{n+1}+b^{n+1}}{a^{n+1}+b^{n+1}}\right)+b\left(\frac{c^{n+1}+a^{n+1}}{c^{n+1}+a^{n+1}}\right)=a+c+b=RHS$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows

$$\sum_{cyc}\left(\frac{a^n(a-b)(a-c)}{b^{n+1}+c^{n+1}}+\frac{a^{n+1}(b+c)}{b^{n+1}+c^{n+1}}\right)\ge a+b+c.$$

Notice that if $a\ge b\ge c$ then

$$\frac{a^n}{b^{n+1}+c^{n+1}}\ge\frac{b^n}{c^{n+1}+a^{n+1}}\ge\frac{c^n}{a^{n+1}+b^{n+1}},$$

so by the Generalized Schur inequality, we deduce that

$$\sum_{cyc}\frac{a^n(a-b)(a-c)}{b^{n+1}+c^{n+1}}\ge 0.$$

So it suffices to prove that

$$\sum_{cyc}\frac{a^{n+1}(b+c)}{b^{n+1}+c^{n+1}}\ge a+b+c\ \Leftrightarrow\ \sum_{cyc}\left(\frac{a^{n+1}(b+c)}{b^{n+1}+c^{n+1}}-a\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{ab(a^n-b^n)}{b^{n+1}+c^{n+1}}-\frac{ca(c^n-a^n)}{b^{n+1}+c^{n+1}}\right)\ge 0\ \Leftrightarrow\ \sum_{cyc}\left(\frac{ab(a^n-b^n)}{b^{n+1}+c^{n+1}}-\frac{ab(a^n-b^n)}{c^{n+1}+a^{n+1}}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{ab(a^n-b^n)(a^{n+1}-b^{n+1})}{(b^{n+1}+c^{n+1})(c^{n+1}+a^{n+1})}\ge 0,$$

which is true because a^n-b^n and $a^{n+1}-b^{n+1}$ have the same sign.

The proof is completed. Equality holds if and only if $a=b=c$.

1325. If $a,b,c>0$ *such that* $a^2+b^2+c^2=3$, then :

$$\frac{a^2(a^2+bc)}{(b+c)^2}+\frac{b^2(b^2+ca)}{(c+a)^2}+\frac{c^2(c^2+ab)}{(a+b)^2}\ge\frac{3}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2(a^2+bc)}{(b+c)^2}+\frac{b^2(b^2+ca)}{(c+a)^2}+\frac{c^2(c^2+ab)}{(a+b)^2}=\sum_{cyc}\frac{a^4}{(b+c)^2}+\sum_{cyc}\frac{a^2bc}{(b+c)^2}$$

$$=\sum_{cyc}\frac{(a^2)^3}{(ab+ac)^2}+abc\sum_{cyc}\frac{a^3}{(ab+ac)^2}\overset{\text{Radon}}{\ge}\frac{\left(\sum_{cyc}a^2\right)^3}{4\left(\sum_{cyc}ab\right)^2}+\frac{abc\left(\sum_{cyc}a\right)^3}{4\left(\sum_{cyc}ab\right)^2}\overset{?}{\ge}\frac{3}{2}$$

$$\overset{a^2+b^2+c^2=3}{\Leftrightarrow}\frac{\left(\sum_{cyc}a^2\right)^3+abc\left(\sum_{cyc}a\right)^3}{4\left(\sum_{cyc}ab\right)^2}\overset{?}{\ge}\frac{1}{2}\sum_{cyc}a^2\overset{a^2+b^2+c^2=3}{\Leftrightarrow}$$

$$\left(\sum_{cyc} a^2\right)^3 + abc\left(\sum_{cyc} a\right)^3 \overset{?}{\underset{(*)}{\geq}} 2\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius
$= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1)$
$\Rightarrow a = s-x, b = s-y, c = s-z \Rightarrow abc = r^2 s \rightarrow (2)$ and substitutions $\Rightarrow$

$$\sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (2)$$

$$\Rightarrow \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab = s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (3) \therefore (1),(2),(3) \Rightarrow (*)$$

$$\Leftrightarrow (s^2 - 8Rr - 2r^2)^3 + r^2 s.s^3 \geq 2(s^2 - 8Rr - 2r^2)(4Rr + r^2)^2$$

$$\Leftrightarrow \boxed{s^6 - (24Rr + 5r^2)s^4 + r^2 s^2(160R^2 + 80Rr + 10r^2) - 4(4Rr + r^2)^3 \overset{(**)}{\geq} 0}$$

and $\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$, it suffices to prove :

$$\text{LHS of } (**) \geq (s^2 - 16Rr + 5r^2)^3 \Leftrightarrow$$

$$(24Rr - 20r^2)s^4 - r^2 s^2(608R^2 - 560Rr + 65r^2)$$

$$+r^3(3840R^3 - 4032R^2 r + 1152Rr^2 - 129r^3) \overset{(***)}{\geq} 0$$

and $\because (24Rr - 20r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(***)$,
it suffices to prove : LHS of $(***) \geq (24Rr - 20r^2)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow \boxed{(160R^2 - 320Rr + 135r^2)s^2 \overset{(****)}{\geq} r(2304R^3 - 4928R^2 r + 2648Rr^2 - 371r^3)} \text{ and}$$

$\because s^2 \overset{\text{Gerretsen}}{\geq} 16Rr - 5r^2$ and $160R^2 - 320Rr + 135r^2 \overset{\text{Euler}}{\geq} 135r^2 > 0$
$\therefore$ in order to prove $(****)$, it suffices to prove :

$$(160R^2 - 320Rr + 135r^2)(16Rr - 5r^2)$$

$$\geq r(2304R^3 - 4928R^2 r + 2648Rr^2 - 371r^3) \Leftrightarrow 32t^3 - 124t^2 + 139t - 38 \geq 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(32t(t-2) + 4t + 19) \geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***)$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{a^2(a^2+bc)}{(b+c)^2} + \frac{b^2(b^2+ca)}{(c+a)^2} + \frac{c^2(c^2+ab)}{(a+b)^2} \geq \frac{3}{2}$$

$$\forall a, b, c > 0 \,|\, a^2 + b^2 + c^2 = 3, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2\left(\frac{a^2(a^2+bc)}{(b+c)^2} + \frac{b^2(b^2+ca)}{(c+a)^2} + \frac{c^2(c^2+ab)}{(a+b)^2}\right) - (a^2 + b^2 + c^2)$$

$$=\sum_{cyc}\left(\frac{2a^2(a^2+bc)}{(b+c)^2}-a^2\right)=\sum_{cyc}\left(\frac{a^2(a^2-b^2)}{(b+c)^2}-\frac{a^2(c^2-a^2)}{(b+c)^2}\right)$$

$$=\sum_{cyc}\left(\frac{a^2(a^2-b^2)}{(b+c)^2}-\frac{b^2(a^2-b^2)}{(c+a)^2}\right)=\sum_{cyc}\frac{(a^2-b^2)[(ca+a^2)^2-(b^2+bc)^2]}{(b+c)^2(c+a)^2}$$

$$=\sum_{cyc}\frac{(a+b)(a+b+c)(ca+a^2+b^2+bc)(a-b)^2}{(b+c)^2(c+a)^2}\geq 0.$$

Therefore,

$$\frac{a^2(a^2+bc)}{(b+c)^2}+\frac{b^2(b^2+ca)}{(c+a)^2}+\frac{c^2(c^2+ab)}{(a+b)^2}\geq\frac{a^2+b^2+c^2}{2}=\frac{3}{2}.$$

Equality holds if and only if $a=b=c=1$.

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{a^4}{(b+c)^2}\geq\sum\frac{a^2(3-a^2)}{2(b+c)^2}\overset{\sum a^2=3\Leftrightarrow 3-a^2=b^2+c^2}{=}\sum\frac{a^2(b^2+c^2)}{2(b+c)^2}$$

$$\frac{1}{2}\cdot\sum\left(\frac{2a^4-a^2\cdot(b^2+c^2)}{(b+c)^2}\right)\overset{?}{\geq}0$$

$$\frac{1}{2}\sum\frac{a^2}{(b+c)^2}[2a^2-(b+c)^2+2bc]\geq 0$$

$$\frac{1}{2}\sum\frac{2(a^2+bc)a^2}{(b+c)^2}-\frac{1}{2}\sum\frac{a^2}{(b+c)^2}\cdot(b+c)^2\geq 0$$

$$\sum\frac{a^2(a^2+bc)}{(b+c)^2}\geq\sum\frac{a^2}{2}=\frac{3}{2}$$

$$a=b=c=1$$

Solution 4 by Sidi Abdellah Lemrabott-Mauritania

This inequality is symmetric assume that $(a\geq b\geq c)$ then:

$\frac{a(a^2+bc)}{b+c}\geq\frac{b(b^2+ca)}{c+a}\geq\frac{c(c^2+ab)}{a+b}$ **and** $\frac{a}{b+c}\geq\frac{b}{c+a}\geq\frac{c}{a+b}$**:**

$$\sum_{cyc}\frac{a^2(a^2+bc)}{(b+c)^2}\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum_{cyc}\frac{a}{b+c}\right)\left(\sum_{cyc}\frac{a(a^2+bc)}{b+c}\right)\overset{Nesbitt}{\geq}$$

$$\geq \frac{1}{2}\left(\sum_{cyc}\frac{a(a^2+bc)}{b+c}\right) \overset{?}{\geq} \frac{3}{2}$$

$$\sum_{cyc}\frac{a(a^2+bc)}{b+c} = \sum_{cyc}\left(\frac{a(a^2+bc)}{b+c} - a^2\right) + \sum_{cyc} a^2 =$$

$$= \sum_{cyc}\frac{a(a-b)(a-c)}{b+c} + \sum_{cyc} a^2 \overset{Schur}{\geq} \sum_{cyc} a^2$$

$$\Leftrightarrow \frac{1}{2}\cdot\left(\sum_{cyc}\frac{a(a^2+bc)}{b+c}\right) \geq \frac{1}{2}\cdot\left(\sum_{cyc} a^2\right) = \frac{3}{2}$$

Equality holds if: $(a=b=c=1)$ (Q.E.D.)

1326. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n; n \in \mathbb{N}.\text{Prove that:}$$

$$\sqrt[n]{a_1^2 + a_2 a_3} + \sqrt[n]{a_2^2 + a_3 a_4} + \cdots + \sqrt[n]{a_n^2 + a_1 a_2} \leq n\sqrt[n]{2}$$

Proposed by Sidi Abdellah Lemrabott-Mauritania

Solution 1 by Elsen Kerimov-Azerbaijan

$$\sqrt[n]{a_1^2 + a_2 a_3} = x_1 \to x_1^n = a_1^2 + a_2 a_3$$

$$\ldots$$

$$\sqrt[n]{a_n^2 + a_1 a_2} = x_n \to x_n^n = a_n^2 + a_1 a_2$$

$$M = x_1^n + x_2^n + \cdots + x_n^n = n + (a_2 a_3 + a_3 a_4 + \cdots + a_1 a_2)$$

$$\left(\frac{1}{n}\sum_{k=1}^{n} x_k^n\right)^{\frac{1}{n}} \geq \frac{1}{n}\left(\sum_{k=1}^{n} x_k\right) \Rightarrow Lemma$$

$$M \geq \frac{(x_1 + x_2 + \cdots + x_n)^n}{n^{n-1}}$$

$$M \leq n + \left(\frac{a_2^2 + a_3^2 + a_3^2 + a_4^2 + \cdots + a_1^2 + a_2^2}{2}\right) = 2n$$

$$\sum_{i=1}^{n} x_i = P$$

$$2n \ge \frac{P^n}{n^{n-1}} \Rightarrow 2n^n \ge P^n \Rightarrow P \le \sqrt[n]{2} \times n$$

Solution 2 by Tapas Das-India

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n$$

$$\sqrt[n]{a_1^2 + a_2a_3} + \sqrt[n]{a_2^2 + a_3a_4} + \cdots + \sqrt[n]{a_n^2 + a_1a_2}$$

$$\overset{CBS}{\le} n\left[\frac{(a_1^2 + a_2^2 + \cdots + a_n^2) + (a_2a_3 + a_3a_4 + \cdots + a_1a_2)}{n}\right]^{\frac{1}{n}}$$

$$\overset{AM-GM}{\le} n\left[\frac{\left(\sum a_1^2\right) + \left(\frac{a_2^2 + a_3^2}{2} + \frac{a_3^2 + a_4^2}{2} + \cdots + \frac{a_1^2 + a_2^2}{2}\right)}{n}\right]^{\frac{1}{n}}$$

$$= n\left[\frac{\left(\sum a_1^2\right) + \left(\sum a_1^2\right)}{n}\right]^{\frac{1}{n}} = n\left[\frac{2\left(\sum a_1^2\right)}{n}\right]^{\frac{1}{n}} = n\left[\frac{2 \cdot n}{n}\right]^{\frac{1}{n}} = n \cdot 2^{\frac{1}{n}} = n\sqrt[n]{2}$$

$$\left[\because \sum a_1^2 = n\right]$$

1327. Let $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{1}{a(a+b)} + \frac{1}{b(b+c)} + \frac{1}{c(c+a)} \ge \frac{a+b+c}{ab+bc+ca-1}.$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc} \frac{1}{b(b+c)} \overset{CBS}{\ge} \frac{\left(\sum_{cyc}\sqrt{\frac{a}{b}}\right)^2}{\sum_{cyc} a(b+c)} = \frac{\sum_{cyc}\left(\frac{a}{b} + 2\sqrt{\frac{a}{c}}\right)}{2(ab+bc+ca)} \overset{AM-GM}{\ge}$$

$$\ge \frac{\sum_{cyc} 3\sqrt[3]{\frac{a}{b} \cdot \sqrt{\frac{a}{c}}^{\,2}}}{2(ab+bc+ca)} = \frac{3(a+b+c)}{2(ab+bc+ca)} =$$

$$=\frac{a+b+c}{ab+bc+ca-\frac{ab+bc+ca}{3}}\overset{AM-GM}{\geq}\frac{a+b+c}{ab+bc+ca-\sqrt[3]{(abc)^2}}=\frac{a+b+c}{ab+bc+ca-1}.$$

Equality holds if and only if $a=b=c=1$.

1328. Let $a,b,c\geq 0$ such that $a+b+c=3$. Prove that :

$$28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)-3\geq 27(ab+bc+ca)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$

$(a=3)$ and then : $LHS-RHS=28*3^{\frac{2023}{2024}}-3>3^3*3^{\frac{2023}{2024}}-3>0$

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $(b+c=3)$

$$\text{and then : } LHS-RHS=28.\left(\sqrt[2024]{\underbrace{b*b*b*\ldots b}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{c*c*c*\ldots c}_{2023\text{ terms}}*1}\right)$$

$$-3-\frac{27.9bc}{(b+c)^2}\overset{G-H}{\geq}28\left(\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}\right)-3-\frac{243bc}{(b+c)^2}$$

$$=28.2024\left(\frac{b^2}{2023b+b^2}+\frac{c^2}{2023c+c^2}\right)-3-\frac{243bc}{(b+c)^2}$$

$$\underset{\because b+c=3}{\overset{\text{Bergstrom and}}{\geq}}\frac{28.2024(b+c)^2}{\frac{2023}{3}(b+c)^2+b^2+c^2}-3-\frac{243bc}{(b+c)^2}$$

$$=\frac{(84.2024-6069)(b^2+c^2+2bc)-9(b^2+c^2)}{2026(b^2+c^2)+2023.2bc}-\frac{243bc}{b^2+c^2+2bc}$$

$$=\frac{163938x+163947y}{2026x+2023y}-\frac{243y}{2(x+y)}\ (x=b^2+c^2;y=2bc)$$

$$=\frac{327876x^2+163452xy-163695y^2}{2(x+y)(2026x+2023y)}$$

$$=\frac{(327876x+491328y)(x-y)+327633y^2}{2(x+y)(2026x+2023y)}>0\because x=b^2+c^2\geq 2bc=y$$

and $b,c>0\Rightarrow x,y>0\therefore 28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)-3>27(ab+bc+ca)$

$\boxed{\text{Case 3}}$ $a,b,c>0$ and then : $LHS-RHS\overset{\because a+b+c=3}{=}$

$$28.\left(\sqrt[2024]{\underbrace{a*a*a*\ldots a}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{b*b*b*\ldots b}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{c*c*c*\ldots c}_{2023\text{ terms}}*1}\right)-3$$

$$-\frac{27.9(ab+bc+ca)}{(a+b+c)^2}\overset{G-H}{\geq}28\left(\frac{2024}{\frac{2023}{a}+1}+\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=28.2024\left(\frac{a^2}{2023a+a^2}+\frac{b^2}{2023b+b^2}+\frac{c^2}{2023c+c^2}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because a+b+c=3}}{\geq}\frac{28.2024(a+b+c)^2}{\frac{2023}{3}(a+b+c)^2+a^2+b^2+c^2}-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{84.2024(u+2v)}{2023(u+2v)+3u}-3-\frac{243v}{u+2v}\ (u=a^2+b^2+c^2, v=ab+bc+ca)$$

$$=\frac{6(27323u^2+27242uv-54565v^2)}{2026u+4046v}=\frac{6(u-v)(27323u+54565v)}{2026u+4046v}\geq 0$$

$$(\because a^2+b^2+c^2\geq ab+bc+ca\Rightarrow u\geq v \text{ and } a,b,c>0\Rightarrow u,v>0)$$

$$\therefore 28\left(\sum_{cyc}a^{\frac{2023}{2024}}\right)-3\geq 27\left(\sum_{cyc}ab\right) \text{ and combining } all \text{ cases,}$$

$$28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)-3\geq 27(ab+bc+ca)$$

$$\forall a,b,c\geq 0 \mid a+b+c=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1329. Let $a, b, c \geq 0$ such that $a+b+c=3$. Prove that :

$$a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}+1\geq ab+bc+ca+a^{\frac{2024}{2025}}b^{\frac{2024}{2025}}c^{\frac{2024}{2025}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$

$$(a=3) \text{ and then : LHS}-\text{RHS}=3^{\frac{2024}{2025}}+1>0$$

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$

$$(b+c=3) \text{ and then : LHS}-\text{RHS}\overset{\because b+c=3}{=}$$

$$\sqrt[2025]{\underbrace{b*b*b*\dots b}_{2024\text{ terms}}*1}+\sqrt[2025]{\underbrace{c*c*c*\dots c}_{2024\text{ terms}}*1}+1-\frac{9bc}{(b+c)^2}$$

$$\overset{G-H}{\geq}\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}+1-\frac{9bc}{(b+c)^2}$$

$$=2025\left(\frac{b^2}{2024b+b^2}+\frac{c^2}{2024c+c^2}\right)+1-\frac{9bc}{(b+c)^2}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because b+c=3}}{\geq}$$

$$\frac{2025(b+c)^2}{\frac{2024}{3}(b+c)^2+b^2+c^2}+1-\frac{9bc}{(b+c)^2}$$

$$=\frac{6075(u+v)}{2027u+2024v}-\frac{9v}{2(u+v)}+1\ (u=b^2+c^2;v=2bc)$$

$$=9*\frac{1350(u+v)^2-v(2027u+2024v)}{2(u+v)(2027u+2024v)}+1=9*\frac{1350u^2+673uv-674v^2}{2(u+v)(2027u+2024v)}+1$$

$$=9*\frac{(u-v)(1350u+2023v)+1349v^2}{2(u+v)(2027u+2024v)}+1\geq 1>0$$

$$\left(\because b^2+c^2\geq 2bc\Rightarrow u\geq v\ and\ b,c>0\Rightarrow u,v>0\right)$$

$$\therefore a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}+1>ab+bc+ca+a^{\frac{2024}{2025}}b^{\frac{2024}{2025}}c^{\frac{2024}{2025}}$$

$\boxed{Case\ 3}$ $a,b,c>0$ and then : LHS − RHS $\overset{\because a+b+c=3}{=}$

$$\sqrt[2025]{\underbrace{a*a*a*\dots a}_{2024\ terms}*1}+\sqrt[2025]{\underbrace{b*b*b*\dots b}_{2024\ terms}*1}+\sqrt[2025]{\underbrace{c*c*c*\dots c}_{2024\ terms}*1}$$
$$+1-\frac{9(ab+bc+ca)}{(a+b+c)^2}-(abc)^{\frac{2024}{2025}}$$

$$\geq\sqrt[2025]{\underbrace{a*a*a*\dots a}_{2024\ terms}*1}+\sqrt[2025]{\underbrace{b*b*b*\dots b}_{2024\ terms}*1}+\sqrt[2025]{\underbrace{c*c*c*\dots c}_{2024\ terms}*1}$$
$$+1-\frac{9(ab+bc+ca)}{(a+b+c)^2}-1$$

$$\left(\because 3=a+b+c\overset{A-G}{\geq}3\sqrt[3]{abc}\Rightarrow abc\leq 1\therefore (abc)^{\frac{2024}{2025}}\leq 1\right)$$

$$\overset{G-H}{\geq}\frac{2025}{\frac{2024}{a}+1}+\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}-\frac{9(ab+bc+ca)}{(a+b+c)^2}$$

$$=2025\left(\frac{a^2}{2024a+a^2}+\frac{b^2}{2024b+b^2}+\frac{c^2}{2024c+c^2}\right)-\frac{9(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{\substack{Bergstrom\\and\\\because a+b+c=3}}{\geq}\frac{2025(a+b+c)^2}{\frac{2024}{3}(a+b+c)^2+a^2+b^2+c^2}-\frac{9(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{6075(x+2y)}{2024(x+2y)+3x}-\frac{9y}{x+2y}\ (x=a^2+b^2+c^2,y=ab+bc+ca)$$

$$=\frac{6075(x+2y)^2-9y(2027x+4048y)}{(x+2y)(2027x+4048y)}=\frac{9(675x^2+673xy-1348y^2)}{(x+2y)(2027x+4048y)}$$

$$=\frac{9(x-y)(675x+1348y)}{(x+2y)(2027x+4048y)}\geq 0$$

$$\left(\because\sum_{cyc}a^2\geq\sum_{cyc}ab\Rightarrow x\geq y\ and\ a,b,c>0\Rightarrow x,y>0\right)$$

$$\therefore\sum_{cyc}a^{\frac{2024}{2025}}+1\geq\sum_{cyc}ab+a^{\frac{2024}{2025}}b^{\frac{2024}{2025}}c^{\frac{2024}{2025}}\ and\ combining\ all\ cases,\sum_{cyc}a^{\frac{2024}{2025}}+1$$
$$\geq\sum_{cyc}ab+a^{\frac{2024}{2025}}b^{\frac{2024}{2025}}c^{\frac{2024}{2025}}\ \forall\ a,b,c\geq 0\mid\sum_{cyc}a=3,''=''\ \text{iff}\ a=b=c=1\ (QED)$$

1330. If $x,y,z>0$ then

$$\sum_{cyc}\frac{x^2}{\left(x+\sqrt{(x+y)(x+z)}\right)^2}\le\frac{x^2y^2+y^2z^2+z^2x^2}{(xy+yz+zx)^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{x^2}{\left(x+\sqrt{(x+y)(x+z)}\right)^2}=\sum_{cyc}\frac{x^2\left(\sqrt{(x+y)(x+z)}-x\right)^2}{(xy+yz+zx)^2}$$

$$\overset{AM-GM}{\le}\frac{\sum_{cyc}x^2\left(\frac{(x+y)+(x+z)}{2}-x\right)^2}{(xy+yz+zx)^2}=\frac{\sum_{cyc}x^2\left(\frac{y+z}{2}\right)^2}{(xy+yz+zx)^2}$$

$$\overset{AM-QM}{\le}\frac{\sum_{cyc}x^2.\frac{y^2+z^2}{2}}{(xy+yz+zx)^2}=\frac{x^2y^2+y^2z^2+z^2x^2}{(xy+yz+zx)^2}.$$

Equality holds iff $x=y=z$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$x+\sqrt{(x+y)(x+z)}=x+\frac{\sqrt{(x+y)(x+z)}}{2}+\frac{\sqrt{(x+y)(x+z)}}{2}\overset{A-G}{\ge}$$

$$3\sqrt[3]{\frac{2x(x+y)(x+z)}{8}}\overset{G-H}{\ge}\frac{3}{2}*\frac{3*2x(x+y)(x+z)}{2x(x+y)+2x(x+z)+(x+y)(x+z)}$$

$$\Rightarrow\frac{x^2}{\left(x+\sqrt{(x+y)(x+z)}\right)^2}$$

$$\le\frac{\left(2x(x+y)(y+z)+2x(x+z)(y+z)+(x+y)(x+z)(y+z)\right)^2}{81(x+y)^2(x+z)^2(y+z)^2}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{x^2}{\left(x+\sqrt{(x+y)(x+z)}\right)^2}\overset{(*)}{\le}$$

$$\sum_{cyc}\frac{\left(2x(x+y)(y+z)+2x(x+z)(y+z)+(x+y)(x+z)(y+z)\right)^2}{81(x+y)^2(y+z)^2(z+x)^2}$$

Assigning $y+z=a,z+x=b,x+y=c\Rightarrow a+b-c=2z>0,b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c,b+c>a,c+a>b\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(\bullet)}{=}s\Rightarrow x=s-a,y=s-b,z=s-c$

Via such substitutions, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)=4Rr+r^2$

$$\Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet)}{=} 4Rr + r^2 \Rightarrow \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x \overset{via\ (\bullet),(\bullet\bullet)}{=}$$

$$(4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet)}{=} r^2\left((4R+r)^2 - 2s^2\right)$$

$$\therefore \sum_{cyc}\left(2x(x+y)(y+z) + 2x(x+z)(y+z) + (x+y)(x+z)(y+z)\right)^2$$

$$= \sum_{cyc}(2(s-a)(ac+ab) + abc)^2 = \sum_{cyc}\left(2(s-a)\left(\sum_{cyc} ab - bc\right) + 4Rrs\right)^2$$

$$= 4\sum_{cyc}\left((s-a) * \sum_{cyc} ab - sbc + 6Rrs\right)^2$$

$$= 4\left(\begin{array}{c}\left(\sum_{cyc} ab\right)^2 * \sum_{cyc}(s-a)^2 + s^2\sum_{cyc} a^2b^2 + 108R^2r^2s^2 - \\ 2s\left(\sum_{cyc} ab\right) * \sum_{cyc} bc(s-a) - 12Rrs^2\sum_{cyc} ab + 12Rrs\left(\sum_{cyc} ab\right) * \sum_{cyc}(s-a)\end{array}\right)$$

$$= 4\left(\begin{array}{c}\left(\sum_{cyc} ab\right)^2 * \left(3s^2 - 2s(2s) + 2(s^2 - 4Rr - r^2)\right) + s^2\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right) \\ +108R^2r^2s^2 - 2s\left(\sum_{cyc} ab\right) * \left(s\left(\sum_{cyc} ab\right) - 12Rrs\right) - 12Rrs^2\sum_{cyc} ab + 12Rrs^2\sum_{cyc} ab\end{array}\right)$$

$$\overset{\sum_{cyc} ab = s^2+4Rr+r^2}{=} 8r^2\left((70R^2 - 4Rr - 2r^2)s^2 - s^4 - r(4R+r)^3\right)$$

$$\therefore \sum_{cyc}\frac{\left(2x(x+y)(y+z) + 2x(x+z)(y+z) + (x+y)(x+z)(y+z)\right)^2}{81(x+y)^2(y+z)^2(z+x)^2}$$

$$\le \frac{8r^2\left((70R^2 - 4Rr - 2r^2)s^2 - s^4 - r(4R+r)^3\right)}{81 * 16R^2r^2s^2} \overset{?}{\le} \frac{\sum_{cyc} x^2y^2}{\left(\sum_{cyc} xy\right)^2} \overset{via\ (\bullet\bullet),(\bullet\bullet\bullet)}{=} \frac{r^2\left((4R+r)^2 - 2s^2\right)}{(4Rr + r^2)^2}$$

$$\Leftrightarrow (1472R^4 + 800R^3r + 156R^2r^2 + 20Rr^3 + 2r^4)s^2$$

$$+r(4R+r)^5 \underset{(**)}{\overset{?}{\ge}} (308R^2 - 8Rr - r^2)s^4$$

Now, RHS of $(**) \overset{Gerretsen}{\le} (308R^2 - 8Rr - r^2)(4R^2 + 4Rr + 3r^2)s^2 \overset{?}{\le}$
$(1472R^4 + 800R^3r + 156R^2r^2 + 20Rr^3 + 2r^4)s^2 + r(4R+r)^5$

$$\Leftrightarrow (240R^4 - 400R^3r - 732R^2r^2 + 48Rr^3 + 5r^4)s^2 + r(4R+r)^5 \underset{(***)}{\overset{?}{\ge}} 0$$

Case 1 $240R^4-400R^3r-732R^2r^2+48Rr^3+5r^4\geq 0$ and then :
LHS of $(***)\geq r(4R+r)^5>0\Rightarrow(***)$ is true (strict inequality)
Case 2 $240R^4-400R^3r-732R^2r^2+48Rr^3+5r^4<0$ and then : LHS of $(***)$
$$=-\left(-\left(240R^4-400R^3r-732R^2r^2+48Rr^3+5r^4\right)\right)s^2+r(4R+r)^5\overset{\text{Gerretsen}}{\geq}$$
$$-\left(-\left(240R^4-400R^3r-732R^2r^2+48Rr^3+5r^4\right)\right)\left(4R^2+4Rr+3r^2\right)$$
$$+r(4R+r)^5\overset{?}{\geq}0$$
$$\Leftrightarrow 120t^6+48t^5-316t^4-412t^3-228t^2+23t+2\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t-2)\left(120t^5+288t^4+260t^3+102t^2+6t(t-2)-1\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$
$\Rightarrow(***)$ is true and combining both cases, $(***)\Rightarrow(**)$ is true $\forall\,\Delta\,ABC$
$$\therefore\sum_{cyc}\frac{x^2}{\left(x+\sqrt{(x+y)(x+z)}\right)^2}\overset{\text{via }(*)}{\leq}$$
$$\sum_{cyc}\frac{\left(2x(x+y)(y+z)+2x(x+z)(y+z)+(x+y)(x+z)(y+z)\right)^2}{81(x+y)^2(y+z)^2(z+x)^2}$$
$$\leq\frac{\sum_{cyc}x^2y^2}{\left(\sum_{cyc}xy\right)^2},\text{''}=\text{''}\text{ iff }x=y=z\text{ (QED)}$$

1331. Let $a,b,c\geq 0$ such that $ab+bc+ca=3$. Prove that
$$\sqrt{\frac{b+c}{bc+1}}+\sqrt{\frac{c+a}{ca+1}}+\sqrt{\frac{a+b}{ab+1}}\geq\frac{a+b+c+3}{\sqrt{a+b+c+abc}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have
$$\frac{(b+c)(a+b+c+abc)}{bc+1}=\frac{(b+c)^2}{bc+1}+\frac{(ab+ca)^2}{ab+ca}\geq\frac{(b+c+ab+ca)^2}{bc+1+ab+ac}$$
$$=\frac{(b+c+ab+ca)^2}{4}\Rightarrow\sqrt{\frac{b+c}{bc+1}}\geq\frac{b+c+ab+ca}{2\sqrt{a+b+c+abc}}.$$
Similarly, we have
$$\sqrt{\frac{c+a}{ca+1}}\geq\frac{c+a+ab+bc}{2\sqrt{a+b+c+abc}}\text{ and }\sqrt{\frac{a+b}{ab+1}}\geq\frac{a+b+bc+ca}{2\sqrt{a+b+c+abc}}.$$
Adding these inequalities, we obtain
$$\sqrt{\frac{b+c}{bc+1}}+\sqrt{\frac{c+a}{ca+1}}+\sqrt{\frac{a+b}{ab+1}}\geq\frac{a+b+c+ab+bc+ca}{\sqrt{a+b+c+abc}}=\frac{a+b+c+3}{\sqrt{a+b+c+abc}}.$$
Equality holds iff $a=b=c=1$.

1332. Let $\lambda \geq 0$ be fixed. If $a, b, c > 0$ such that : $abc = 1$, then :

$$\sum_{cyc} \frac{a^2}{(ab+\lambda)(\lambda ab+1)} \geq \frac{3}{(\lambda+1)^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$xy \leq \frac{(x+y)^2}{4} \therefore \sum_{cyc} \frac{a^2}{(ab+\lambda)(\lambda ab+1)} \geq \sum_{cyc} \frac{a^2}{\frac{(ab+\lambda+\lambda ab+1)^2}{4}}$$

$$= 4\sum_{cyc} \frac{a^2}{(\lambda+1)^2(ab+1)^2} \geq \frac{4}{(\lambda+1)^2} \sum_{cyc} \frac{ab}{(ab+1)(bc+1)}$$

$$(\because x^2+y^2+z^2 \geq xy+yz+zx)$$

$$= \frac{4}{(\lambda+1)^2(ab+1)(bc+1)(ca+1)} \sum_{cyc} ab(ca+1) \overset{abc=1}{=}$$

$$\frac{4\left(\sum_{cyc} a + \sum_{cyc} ab\right)}{(\lambda+1)^2\left(2+\sum_{cyc} a + \sum_{cyc} ab\right)} \overset{?}{\geq} \frac{3}{(\lambda+1)^2}$$

$$\Leftrightarrow 4\sum_{cyc} a + 4\sum_{cyc} ab \overset{?}{\geq} 6 + 3\sum_{cyc} a + 3\sum_{cyc} ab \Leftrightarrow \sum_{cyc} a + \sum_{cyc} ab \overset{?}{\geq} 6$$

$$\rightarrow \text{true} \because \sum_{cyc} a + \sum_{cyc} ab \overset{A-G}{\geq} 3 * \sqrt[3]{abc} + 3 * \sqrt[3]{a^2b^2c^2} \overset{abc=1}{=} 6$$

$$\therefore \sum_{cyc} \frac{a^2}{(ab+\lambda)(\lambda ab+1)} \geq \frac{3}{(\lambda+1)^2} \ \forall\, a,b,c > 0 \,\big|\, abc = 1 \text{ and } \forall\, \lambda \geq 0,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a = \frac{x}{y}, b = \frac{z}{x}, c = \frac{y}{z}$, where $x, y, z > 0$. We have

$$\sum_{cyc} \frac{a^2}{(ab+\lambda)(\lambda ab+1)} = \sum_{cyc} \frac{x^2}{(z+\lambda y)(\lambda z+y)} \overset{Hölder}{\geq} \frac{\left(\sum_{cyc} x\right)^3}{\sum_{cyc} x(z+\lambda y) . \sum_{cyc}(\lambda z+y)}$$

$$= \frac{\left(\sum_{cyc} x\right)^2}{(\lambda+1)^2 \sum_{cyc} xy} \geq \frac{3}{(\lambda+1)^2}.$$

Equality holds iff $a = b = c = 1$.

1333. Let $a, b, c > 0$ such that $a^2+b^2+c^2 = a+b+c$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+3 \geq \frac{\sqrt{5a^2+4bc}-a}{\sqrt{bc}} + \frac{\sqrt{5b^2+4ca}-b}{\sqrt{ca}} + \frac{\sqrt{5c^2+4ab}-c}{\sqrt{ab}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By the AM − GM inequality, we have

$$\frac{\sqrt{5a^2+4bc}-a}{\sqrt{bc}}=\sqrt{\frac{5a^2}{bc}+4}-\frac{a}{\sqrt{bc}}\le\frac{1}{2}\left(\frac{\frac{5a^2}{bc}+4}{\frac{a}{\sqrt{bc}}+2}+\frac{a}{\sqrt{bc}}+2\right)-\frac{a}{\sqrt{bc}}=\frac{\frac{2a^2}{bc}+4}{\frac{a}{\sqrt{bc}}+2}$$

$$\le\frac{\frac{2a^2}{bc}+4}{\frac{2a}{b+c}+2}=\frac{a^2\left(\frac{1}{b}+\frac{1}{c}\right)+2(b+c)}{a+b+c}=\frac{a^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-a+2b+2c}{a+b+c}\quad\text{(and analogs)}$$

Therefore

$$\sum_{cyc}\frac{\sqrt{5a^2+4bc}-a}{\sqrt{bc}}\le\frac{(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+3(a+b+c)}{a+b+c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+3.$$

Equality holds if and only if $a=b=c=1$.

1334. Let $a,b,c>0$ such that $abc\ge 1$ then prove that

$$\frac{a^5(b^6+c^6)}{a^6+b^5c}+\frac{b^5(c^6+a^6)}{b^6+c^5a}+\frac{c^5(a^6+b^6)}{c^6+a^5b}\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$a^6+b^5c\le\sqrt[6]{(a^6+b^6)^5(a^6+c^6)}\quad\text{(and analogs)}$$

Then by using this result and AM − GM inequality, we get

$$\sum_{cyc}\frac{a^5(b^6+c^6)}{a^6+b^5c}\ge\sum_{cyc}\frac{a^5(b^6+c^6)}{\sqrt[6]{(a^6+b^6)^5(a^6+c^6)}}\ge 3\sqrt[3]{\prod_{cyc}\frac{a^5(b^6+c^6)}{\sqrt[6]{(a^6+b^6)^5(a^6+c^6)}}}$$

$$=3\sqrt[3]{(abc)^5}\ge 3.$$

Equality holds iff $a=b=c=1$.

1335. Let $a,b,c>0$ such that $a+b+c+2=abc$. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 2\left(\sqrt{\frac{1}{a^2}+\frac{a(b+c)}{bc}}+\sqrt{\frac{1}{b^2}+\frac{b(c+a)}{ca}}+\sqrt{\frac{1}{c^2}+\frac{c(a+b)}{ab}}\right)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$2\sqrt{\frac{1}{a^2}+\frac{a(b+c)}{bc}}\le\left(\frac{1}{a}+1\right)+\frac{\frac{1}{a^2}+\frac{a(b+c)}{bc}}{\frac{1}{a}+1}=\frac{2}{a}+\frac{\frac{ab+bc+ca}{bc}}{\frac{1}{a}+1}$$

$$=\frac{2}{a}+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(a-1+\frac{1}{a+1}\right)\ \text{(and analogs)}$$

Then

$$2\sum_{cyc}\sqrt{\frac{1}{a^2}+\frac{a(b+c)}{bc}}\le\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(2+a+b+c-3+\sum_{cyc}\frac{1}{a+1}\right)$$

$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c),$$

$$\text{because we have }\sum_{cyc}\frac{1}{a+1}=1.$$

So the proof is complete. Equality holds iff $a=b=c=2$.

1336. Let $a,b,c>0$ such that $abc=1$. Prove that

$$\frac{a^3b+b^3c+c^3a}{a+b+c}+1\ge 2\sqrt{\frac{a^2b^2+b^2c^2+c^2a^2}{a+b+c}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Applying Cauchy – Schwarz inequality twice, we obtain

$$\frac{a^3b+b^3c+c^3a}{a+b+c}+1=\frac{a^3b+b^3c+c^3a+abc(a+b+c)}{a+b+c}$$

$$=\frac{(a^2+b^2+c^2)(ab+bc+ca)-(ab^3+bc^3+ca^3)}{a+b+c}$$

$$=\frac{\sqrt{[(a^4+b^4+c^4)+2(a^2b^2+b^2c^2+c^2a^2)][(a^2b^2+b^2c^2+c^2a^2)+2(a+b+c)]}-(ab^3+bc^3+ca^3)}{a+b+c}$$

$$\ge\frac{\sqrt{(a^4+b^4+c^4)(c^2a^2+a^2b^2+b^2c^2)}+2\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a+b+c)}-(ab^3+bc^3+ca^3)}{a+b+c}$$

$$\ge\frac{(ab^3+bc^3+ca^3)+2\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a+b+c)}-(ab^3+bc^3+ca^3)}{a+b+c}$$

$$=2\sqrt{\frac{a^2b^2+b^2c^2+c^2a^2}{a+b+c}},$$

as desired. Equality holds iff $a=b=c=1$.

1337.

Given positive real numbers that $ab+bc+ca=abc$. Prove that

$$2\sqrt{\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}}\le\frac{1}{4}+\frac{a}{b+ca}+\frac{b}{c+ab}+\frac{c}{a+bc}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{1}{4}+\frac{a}{b+ca}+\frac{b}{c+ab}+\frac{c}{a+bc}=\frac{1}{4}+\frac{ab}{b^2+abc}+\frac{bc}{c^2+abc}+\frac{ca}{a^2+abc}$$

$$\ge 4\sqrt[4]{\frac{1}{4}\cdot\frac{ab}{(b+c)(b+a)}\cdot\frac{bc}{(c+a)(c+b)}\cdot\frac{ca}{(a+b)(a+c)}}=2\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}$$

$$=2\sqrt{\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)}}=2\sqrt{\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}}$$

$$=2\sqrt{\frac{a}{a^2+abc}+\frac{b}{b^2+abc}+\frac{c}{c^2+abc}}=2\sqrt{\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}}.$$

Equality holds iff $a=b=c=3$.

1338. **Let $a,b,c>0$ *such that* : $ab+bc+ca=3$. Prove that :**

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}-\frac{3}{2(a+b+c+abc)}\ge\frac{3}{8}$$

Proposed by Nguyen Thai An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}-\frac{3}{2(a+b+c+abc)}\ge$$

$$\frac{1}{(a+1)(b+1)}+\frac{1}{(b+1)(c+1)}+\frac{1}{(c+1)(a+1)}-\frac{3}{2(a+b+c+abc)}$$

$$=\frac{\sum_{cyc}a+3}{\sum_{cyc}a+\sum_{cyc}ab+abc+1}-\frac{3}{2(\sum_{cyc}a+abc)}\overset{ab+bc+ca=3}{=}$$

$$\frac{\sum_{cyc}a+3}{\sum_{cyc}a+abc+4}-\frac{3}{2(\sum_{cyc}a+abc)}\to(1)$$

$\boxed{\text{Case 1}}\ \sum_{cyc} a + abc \ge 4$ and $\because \sum_{cyc} ab = 3 \therefore 3 \overset{A-G}{\ge} 3\sqrt[3]{a^2b^2c^2} \Rightarrow 1 \ge abc$

$$\Rightarrow \frac{\sum_{cyc} a + 3}{\sum_{cyc} a + abc + 4} - \frac{3}{2(\sum_{cyc} a + abc)} \ge \frac{\sum_{cyc} a + abc + 2}{\sum_{cyc} a + abc + 4} - \frac{3}{2(\sum_{cyc} a + abc)}$$

$$= \frac{x+2}{x+4} - \frac{3}{2x}\left(x = \sum_{cyc} a + abc > 0\right) \overset{?}{\ge} \frac{3}{8}$$

$$\Leftrightarrow \frac{8x(x+2) - 12(x+4) - 3x(x+4)}{8x(x+4)} \overset{?}{\ge} 0 \Leftrightarrow 5x^2 - 8x - 48 \overset{?}{\ge} 0$$

$$\Leftrightarrow (x-4)(5x+12) \overset{?}{\ge} 0 \Leftrightarrow x = \sum_{cyc} a + abc \ge 4 \to \text{true}$$

$$\therefore \frac{\sum_{cyc} a + 3}{\sum_{cyc} a + abc + 4} - \frac{3}{2(\sum_{cyc} a + abc)} \ge \frac{3}{8}$$

$\boxed{\text{Case 2}}\ \sum_{cyc} a + abc \le 4$ and then : $\frac{\sum_{cyc} a + 3}{\sum_{cyc} a + abc + 4} - \frac{3}{2(\sum_{cyc} a + abc)} \ge$

$$\frac{\sum_{cyc} a + 3}{8} - \frac{3}{2(\sum_{cyc} a + abc)} \overset{?}{\ge} \frac{3}{8} \Leftrightarrow \frac{\sum_{cyc} a}{8} \overset{?}{\ge} \frac{3}{2(\sum_{cyc} a + abc)}$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)^2 + abc\left(\sum_{cyc} a\right) \overset{?}{\ge} 12$$

$$\overset{ab+bc+ca=3}{\Leftrightarrow} \frac{(\sum_{cyc} ab)(\sum_{cyc} a)^2}{3} + abc\left(\sum_{cyc} a\right) \overset{?}{\ge} \frac{12}{9}\left(\sum_{cyc} ab\right)^2$$

$$\Leftrightarrow \sum_{cyc} a^3b + \sum_{cyc} ab^3 \overset{?}{\ge} 2\sum_{cyc} a^2b^2 \Leftrightarrow \sum_{cyc} ab(a-b)^2 \overset{?}{\ge} 0 \to \text{true}$$

$$\therefore \frac{\sum_{cyc} a + 3}{\sum_{cyc} a + abc + 4} - \frac{3}{2(\sum_{cyc} a + abc)} \ge \frac{3}{8}$$

$\therefore$ combining both cases, $\frac{\sum_{cyc} a + 3}{\sum_{cyc} a + abc + 4} - \frac{3}{2(\sum_{cyc} a + abc)} \ge \frac{3}{8}$

$$\overset{\text{via }(1)}{\Rightarrow} \frac{1}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{1}{(c+1)^2} - \frac{3}{2(a+b+c+abc)} \ge \frac{3}{8}$$

$\forall a, b, c > 0 \mid ab + bc + ca = 3, '' ='' $ iff $a = b = c = 1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := \frac{1}{a+1},\ y := \frac{1}{b+1},\ z := \frac{1}{c+1}$ and

$p := x + y + z,\ q := xy + yz + zx,\ r := xyz.$

We have $a = \frac{1-x}{x},\ b = \frac{1-y}{y},\ c = \frac{1-z}{z}.$

The given condition is equivalent to $p = 2q$.

Since $(x+y+z)^2 \geq 3(xy+yz+zx)$ then

$$p \geq \frac{3q}{p} = \frac{3}{2}, \text{ and since } (1-x)(1-y)(1-z) \geq 0$$

$$\text{then } 1 \geq p-q+r = \frac{p}{2}+r \geq \frac{p}{2}, \quad \text{so } 2 \geq p \geq \frac{3}{2}.$$

Now the desired inequality is equivalent to

$$x^2+y^2+z^2 \geq \frac{3}{8}+\frac{3}{2\left(\frac{1}{xyz}-4\right)} = \frac{3}{8(1-4xyz)} \quad \text{or} \quad 8(1-4r)(p^2-p) \geq 3 \quad (1)$$

Since $(xy+yz+zx)^2 \geq 3xyz(x+y+z)$ then $r \leq \frac{q^2}{3p} = \frac{p}{12}$, and

$$LHS_{(1)} \geq 8\left(1-\frac{p}{3}\right)(p^2-p) = 3+\frac{(2p-3)[4p(2-p)+2p+3]}{3} \geq 3,$$

So the proof is complete. Equality holds iff $a=b=c=1$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } p := a+b+c \geq 3, \quad q := ab+bc+ca = 3,$$
$$r := abc \leq 1. \text{ By AM} - \text{GM inequality, we have}$$

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2} \geq \frac{1}{(a+1)(b+1)}+\frac{1}{(b+1)(c+1)}+\frac{1}{(c+1)(a+1)}$$
$$= \frac{p+3}{p+r+4}.$$

So it suffices to prove that

$$\frac{p+3}{p+r+4}-\frac{3}{2(p+r)} \geq \frac{3}{8} \quad \text{or} \quad f(r) = 5p^2+2pr-48-3r^2 \geq 0.$$

We have $f'(r) = 2(p-3r) \geq 0$, because $p \geq 3 \geq 3r$, so f is increasing.

- If $p \geq 4$ then $f(r) \geq f(0) = 5p^2-48 \geq 5.4^2-48 > 0$.
- If $3 \leq p \leq 4$ then $f(r) \overset{Schur}{\geq} f\left(\frac{4pq-p^3}{9}\right)$

$$= 5p^2+\frac{2p^2(12-p^2)}{9}-48-\frac{p^2(12-p^2)^2}{27}$$
$$= \frac{18p^4-p^6+63p^2-1296}{27} = \frac{(p^2-9)[(16-p^2)(9+p^2)+2p^2]}{27} \geq 0.$$

So the proof is complete. Equality holds iff $a=b=c=1$.

1339.

Let $a,b,c \geq 0$ such that $2(ab+bc+ca) = a+b+c$. Prove that

$$\frac{a(b+c-1)}{a+1}+\frac{b(c+a-1)}{b+1}+\frac{c(a+b-1)}{c+1} \geq 0$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{a(b+c-1)}{a+1}=\sum_{cyc}\left(\frac{a(a+b+c)}{a+1}-a\right)=(a+b+c)\left(\sum_{cyc}\frac{a}{a+1}-1\right)$$

$$\overset{CBS}{\geq}\sum_{cyc}a.\left(\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a^2+\sum_{cyc}a}-1\right)=\sum_{cyc}a.\left(\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a^2+2\sum_{cyc}bc}-1\right)=0.$$

Equality holds iff

$$a=b=c=\frac{1}{2},a=b=c=0\ or\ a=b=1,c=0\text{ and permutations.}$$

1340. Given nonnegative real numbers that

$$a+b+c=a^2+b^2+c^2,ab+bc+ca>0.\text{ Prove that}$$

$$a\sqrt{\frac{1+b^2+c^2}{b^2+c^2}}+b\sqrt{\frac{1+c^2+a^2}{c^2+a^2}}+c\sqrt{\frac{1+a^2+b^2}{a^2+b^2}}\geq 2\sqrt{a+b+c}$$

Proposed by Nguyen Thai An, Thai Ha Nhat Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{1+b^2+c^2}{b^2+c^2}=\frac{1}{b^2+c^2}+1\geq\frac{(1+a)^2}{(b^2+c^2)+a^2}=\frac{(1+a)^2}{a+b+c}\ \text{(and analogs)}$$

Then

$$\sum_{cyc}a\sqrt{\frac{1+b^2+c^2}{b^2+c^2}}\geq\sum_{cyc}\frac{a(1+a)}{\sqrt{a+b+c}}=\frac{2(a+b+c)}{\sqrt{a+b+c}}=2\sqrt{a+b+c}.$$

Equality holds iff $a=b=1,c=0$ and permutation.

1341. Let $a,b,c\geq 0$ such that $a+b+c=3$. Prove that

$$\sqrt{a^2+a}+\sqrt{b^2+b}+\sqrt{c^2+c}\leq\sqrt{\frac{3\sqrt[3]{abc}+33}{2}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c=3,\ \ q:=ab+bc+ca,$

$r:=abc\leq\left(\frac{p}{3}\right)^3=1.$ **By CBS inequality, we have**

$$\sum_{cyc}\sqrt{a^2+a}\le\sqrt{\sum_{cyc}(3a+1).\sum_{cyc}\frac{a^2+a}{3a+1}}=\sqrt{12\sum_{cyc}\left(\frac{a}{3}+\frac{2}{9}-\frac{2}{9(3a+1)}\right)}$$

$$=\sqrt{12\left(1+\frac{2}{3}-\frac{2(9q+6p+3)}{9(27r+9q+3p+1)}\right)}=\sqrt{\frac{12(12+13q+45r)}{27r+9q+10}}.$$

By AM − GM inequality, we have $\sqrt[3]{abc}=\sqrt[3]{r}\ge\frac{3r}{2r+1}$**, so it suffices to prove that**

$$\frac{12(12+13q+45r)}{27r+9q+10}\le\frac{9r}{2(2r+1)}+\frac{33}{2}\Leftrightarrow 14+(17r-5)q-5r-45r^2\ge 0\ (*)$$

• **If** $17r\ge 5$**. By AM − GM inequality, we have** $pq\ge 9r$ **then** $q\ge 3r$ **and,**

$$LHS_{(*)}\ge 14+(17r-5).3r-5r-45r^2=2(1-r)(7-3r)\ge 0.$$

• **If** $17r\le 5$**. By Schur's inequality, we have** $q\le\frac{p^3+9r}{4p}=\frac{9+3r}{4}$ **then**

$$LHS_{(*)}\ge 14+\frac{(17r-5)(9+3r)}{4}-5r-45r^2=\frac{(1-r)(11+129r)}{4}\ge 0.$$

So the proof is complete. Equality holds iff $a=b=c=1$**.**

1342. If $x,y,z>0,xy+yz+zx=3$ **then**

$$\sum_{cyc}\sqrt[5]{\frac{yz}{4(x^5-x+8)}}\le\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $x^5-x+8=2(x^2+3)+(x^3+2x^2+3x+2)(x-1)^2\ge 2(x^2+3)$**,**
$\forall x>0,$ **then**

$$\sqrt[5]{\frac{yz}{4(x^5-x+8)}}\le\sqrt[5]{\frac{yz}{8(x^2+3)}}$$

$$=\sqrt[5]{\frac{yz}{8(x+y)(x+z)}}\overset{AM-GM}{\le}\frac{1}{5}\left(\frac{y}{x+y}+\frac{z}{z+x}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right).$$

$$\Rightarrow\sqrt[5]{\frac{yz}{4(x^5-x+8)}}\le\frac{1}{5}\left(\frac{y}{x+y}+\frac{z}{z+x}\right)+\frac{3}{10}\ \text{(and analogs)}$$

Therefore

$$\sum_{cyc}\sqrt[5]{\frac{yz}{4(x^5-x+8)}}\le\frac{1}{5}\sum_{cyc}\left(\frac{y}{x+y}+\frac{z}{z+x}\right)+\frac{9}{10}=\frac{1}{5}\sum_{cyc}\left(\frac{y}{x+y}+\frac{x}{x+y}\right)+\frac{9}{10}$$

$$=\frac{1}{5}.3+\frac{9}{10}=\frac{3}{2}.$$

Equality holds iff $x=y=z=1$.

1343. Let $a,b,c\ge 0$ such that $a+b+c=3$. Prove that

$$\sqrt{a(8a+8-\sqrt[3]{abc})}+\sqrt{b(8b+8-\sqrt[3]{abc})}+\sqrt{c(8c+8-\sqrt[3]{abc})}\le 3\sqrt{15}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

If two of the numbers a,b,c are zero then we have $LHS=4\sqrt{6}\le 3\sqrt{15}$.

Assume now that $ab+bc+ca>0$. By AM − GM inequality, we have

$$2\sqrt{15a(8a+8-\sqrt[3]{abc})}\le 5\sqrt[3]{a^2}(2\sqrt[3]{a}+\sqrt[6]{bc})+\frac{3\sqrt[3]{a}.(8a+8-\sqrt[3]{abc})}{2\sqrt[3]{a}+\sqrt[6]{bc}}$$

$$=20a+\frac{2\sqrt[3]{a}.(2a+12+\sqrt[3]{abc})}{2\sqrt[3]{a}+\sqrt[6]{bc}}\le 20a+\frac{2\sqrt[3]{a}.(2a+12+\sqrt[3]{abc})}{2\sqrt[3]{a}+\frac{2\sqrt[3]{bc}}{\sqrt[3]{b}+\sqrt[3]{c}}}$$

$$=20a+\frac{(\sqrt[3]{ab}+\sqrt[3]{ca})(2a+12+\sqrt[3]{abc})}{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}$$

$$=22a+\frac{12(\sqrt[3]{ab}+\sqrt[3]{ca})-(2\sqrt[3]{a^2}-\sqrt[3]{ab}-\sqrt[3]{ca})\sqrt[3]{abc}}{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}.$$

Adding this inequality with the similar ones, we obtain

$$2\sqrt{15}.\sum_{cyc}\sqrt{a(8a+8-\sqrt[3]{abc})}$$

$$\le 22(a+b+c)+12.2-\frac{2(\sum_{cyc}\sqrt[3]{a^2}-\sum_{cyc}\sqrt[3]{bc}).\sqrt[3]{abc}}{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}$$

$$\le 22.3+24-0=90.$$

So the proof is complete. Equality holds iff $a=b=c=1$.

1344. If $a,\mathrm{b},\mathrm{c}>0$ *such that* : $ab+\mathrm{bc}+\mathrm{ca}=3$ *and* $\lambda\ge 1$, then :

$$\sum_{\mathrm{cyc}}\frac{1}{a^2+\mathrm{b}^2+\lambda}\le\frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{a^2+b^2+\lambda}\le\frac{3}{\lambda+2}\Leftrightarrow\sum_{cyc}\left(\frac{1}{\lambda+2}-\frac{1}{a^2+b^2+\lambda}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{a^2+b^2+\lambda-\lambda-2}{(\lambda+2)(a^2+b^2+\lambda)}\Leftrightarrow\frac{x-2}{x+\lambda}+\frac{y-2}{y+\lambda}+\frac{z-2}{z+\lambda}\ge 0$$

$$\left(x=a^2+b^2, y=b^2+c^2, z=c^2+a^2\right)$$

$$\Leftrightarrow\sum_{cyc}(x-2)(y+\lambda)(z+\lambda)\ge 0$$

$$\Leftrightarrow\boxed{3xyz-2\sum_{cyc}xy+2\lambda\sum_{cyc}xy-4\lambda\sum_{cyc}x+\lambda^2\sum_{cyc}x-6\lambda^2\overset{(*)}{\ge}0}$$

$$\text{Now, }2\lambda\sum_{cyc}xy-4\lambda\sum_{cyc}x$$

$$=2\lambda\left(\sum_{cyc}(a^2+b^2)(b^2+c^2)-\frac{2}{3}\left(\sum_{cyc}(a^2+b^2)\right)\left(\sum_{cyc}ab\right)\right)$$

$$=2\lambda\left(\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2-\frac{4}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)\right)$$

$$\ge 2\lambda\left(\left(\sum_{cyc}a^2\right)^2+\frac{\left(\sum_{cyc}ab\right)^2}{3}-\frac{4}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)\right)$$

$$=2\lambda\left(\frac{3m^2+n^2-4mn}{3}\right)\left(m=\sum_{cyc}a^2, n=\sum_{cyc}ab\right)$$

$$=\frac{2\lambda}{3}*(m-n)(3m-n)\overset{\lambda\ge 1\ and\ \because m\ge n}{\ge}\frac{2}{3}*(m-n)(3m-n)$$

$$\therefore\boxed{2\lambda\sum_{cyc}xy-4\lambda\sum_{cyc}x\overset{(\bullet)}{\ge}\frac{2}{3}*(m-n)(3m-n)}$$

$$\text{Again, }3xyz-2\sum_{cyc}xy=3\prod_{cyc}(a^2+b^2)-\frac{2}{3}\left(\sum_{cyc}(a^2+b^2)(b^2+c^2)\right)\left(\sum_{cyc}ab\right)$$

$$=3\left(\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-a^2b^2c^2\right)-\frac{2}{3}\left(\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}ab\right)$$

$$=3\left(\left(\frac{(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)}{9}-a^2b^2c^2\right)+\frac{8}{9}*\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)\right)$$

$$-\frac{2}{3}\left(\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}ab\right)$$

$$\overset{A-G}{\geq}\frac{8}{3}*\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-\frac{2}{3}\left(\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}ab\right)$$

$$=\frac{2}{3}\left(\left(\sum_{cyc}a^2b^2\right)\left(4\sum_{cyc}a^2-\sum_{cyc}ab\right)-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)^2\right)$$

$$\geq\frac{2}{3}\left(\frac{1}{3}\left(\sum_{cyc}ab\right)^2\left(4\sum_{cyc}a^2-\sum_{cyc}ab\right)-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)^2\right)\overset{ab+bc+ca=3}{=}$$

$$\frac{2}{3}\left(n(4m-n)-3m^2\right)=\frac{-2}{3}\left(3m^2+n^2-4mn\right)$$

$$\Rightarrow\boxed{3xyz-2\sum_{cyc}xy\overset{(\bullet\bullet)}{\geq}\frac{-2}{3}*(m-n)(3m-n)}$$

$$\text{Also, }\lambda^2\sum_{cyc}x-6\lambda^2=2\lambda^2\left(\sum_{cyc}a^2\right)-6\lambda^2\geq2\lambda^2\left(\sum_{cyc}ab\right)-6\lambda^2$$

$$\overset{ab+bc+ca=3}{=}6\lambda^2-6\lambda^2\Rightarrow\boxed{\lambda^2\sum_{cyc}x-6\lambda^2\overset{(\bullet\bullet\bullet)}{\geq}0}\therefore(\bullet)+(\bullet\bullet)+(\bullet\bullet\bullet)\Rightarrow$$

$$\text{LHS of }(*)\geq0\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{1}{a^2+b^2+\lambda}\leq\frac{3}{\lambda+2}$$

$$\forall a,b,c>0\,\middle|\,ab+bc+ca=3\text{ and }\lambda\geq1,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

1345. Let $a,b,c\geq0:\ a^2+b^2+c^2+abc=4$. Prove that

$$a\sqrt{a^2+2}+b\sqrt{b^2+2}+c\sqrt{c^2+2}\leq\sqrt{6(a^2+b^2+c^2)+9}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $a^2+b^2+c^2+abc=4$ then $\exists x,y,z\geq0, xy+yz+zx>0$ such that

$$a=\frac{2x}{\sqrt{(x+y)(x+z)}},$$

$b=\frac{2y}{\sqrt{(y+z)(y+x)}}, c=\frac{2z}{\sqrt{(z+x)(z+y)}}$. The desired inequality is equivalent to

$$2\sum_{cyc} x\sqrt{\frac{2(y+z)(3x^2+xy+yz+zx)}{(x+y)(x+z)}} \le \sqrt{33(x+y+z)(xy+yz+zx)-81xyz}.$$

Now let $p := x+y+z$, $q := xy+yz+zx$, $r := xyz$. By CBS inequality, we have

$$\left(LHS_{(1)}\right)^2 \le 8\sum_{cyc} x.\sum_{cyc}\frac{x(y+z)(3x^2+q)}{(x+y)(x+z)} = \frac{8p\left(3\sum_{cyc}x^3(y+z)^2+q\sum_{cyc}x(y+z)^2\right)}{(x+y)(y+z)(z+x)},$$

Since $\sum_{cyc} x^3(y+z)^2 = pq^2-5qr, \sum_{cyc} x(y+z)^2 = pq+3r,$

$(x+y)(y+z)(z+x) = pq-r$, then

$$\left(LHS_{(1)}\right)^2 \le \frac{32pq(pq-3r)}{pq-r} \overset{?}{\lesssim} 33pq-81r \Leftrightarrow (pq-9r)^2 \ge 0,$$

which is true, so the proof is complete. Equality holds iff $a=b=c=1$.

1346. If $a,b,c>0$ *such that* : $ab+bc+ca=1$ and $0\le\lambda\le\frac{2}{\sqrt{3}}$, then :

$$\sum_{cyc}\frac{1}{\lambda+a} \ge \frac{3\sqrt{3}}{1+\lambda\sqrt{3}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} ab\right)^2 \ge 3abc\left(\sum_{cyc} a\right) \overset{ab+bc+ca=1}{\Rightarrow} 3abc\left(\sum_{cyc} a\right) \le 1 \Rightarrow abc \overset{(*)}{\le} \frac{1}{3t}$$

$$\left(t=\sum_{cyc} a, say\right) \text{ and } \left(\sum_{cyc} a\right)^2 \ge 3\left(\sum_{cyc} ab\right) \overset{ab+bc+ca=1}{=} 3 \Rightarrow t-\sqrt{3} \overset{(**)}{\ge} 0$$

$$\text{Now, } \sum_{cyc}\frac{1}{\lambda+a} = \frac{\sum_{cyc}(\lambda^2+\lambda(b+c)+bc)}{\lambda^3+\lambda^2(\sum_{cyc} a)+\lambda(\sum_{cyc} ab)+abc} \overset{\substack{\text{via } (*) \\ \text{and} \\ \because ab+bc+ca=1}}{\ge}$$

$$\frac{3\lambda^2+2\lambda t+1}{\lambda^3+\lambda^2 t+\lambda+\frac{1}{3t}} = \frac{9\lambda^2 t+6\lambda t^2+3t}{3\lambda^3 t+3\lambda^2 t^2+3\lambda t+1} \overset{?}{\ge} \frac{3\sqrt{3}}{1+\lambda\sqrt{3}}$$

$$\Leftrightarrow t+2\lambda t^2+3\lambda^2 t+\lambda\sqrt{3}*t+2\sqrt{3}*\lambda^2 t^2+3\sqrt{3}*\lambda^3 t \overset{?}{\ge}$$
$$3\sqrt{3}*\lambda t+3\sqrt{3}*\lambda^2 t^2+3\sqrt{3}*\lambda^3 t+\sqrt{3}$$

$$\Leftrightarrow t-\sqrt{3}+\lambda*2t(t-\sqrt{3})-\lambda^2*\sqrt{3}t(t-\sqrt{3})\overset{?}{\underset{(\bullet)}{\geq}}0$$

$$\because 0\le\lambda\le\frac{2}{\sqrt{3}}\ \textit{and}\ t-\sqrt{3}\overset{\text{via }(**)}{\geq}0\ \therefore \text{LHS of }(\bullet)\geq$$

$$0+\lambda*2t(t-\sqrt{3})-\lambda*\frac{2}{\sqrt{3}}*\sqrt{3}t(t-\sqrt{3})=0\Rightarrow(\bullet)\text{ is true}$$

$$\Rightarrow\sum_{cyc}\frac{1}{\lambda+a}\geq\frac{3\sqrt{3}}{1+\lambda\sqrt{3}}\ \forall\, a,b,c>0\,\big|\,ab+bc+ca=1\ \textit{and}\ 0\le\lambda\le\frac{2}{\sqrt{3}},$$

$$''=''\ \text{ iff } a=b=c=\frac{1}{\sqrt{3}}\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c,\ \ q:=ab+bc+ca=1,\ \ r:=abc$. We have

$$\sum_{cyc}\frac{1}{\lambda+a}=\frac{3\lambda^2+2\lambda(a+b+c)+(ab+bc+ca)}{\lambda^3+\lambda^2(a+b+c)+\lambda(ab+bc+ca)+abc}$$

$$=\frac{3\lambda^2+2\lambda p+1}{\lambda^3+\lambda^2p+\lambda+r}\overset{?}{\geq}\frac{3\sqrt{3}}{1+\lambda\sqrt{3}}$$

$$\Leftrightarrow 3\lambda^2-2\sqrt{3}\lambda+1+\lambda(2-\sqrt{3}\lambda)p-3\sqrt{3}r\geq 0,$$

Since $(a+b+c)^2\geq 3(ab+bc+ca)$ and $ab+bc+ca\geq 3\sqrt[3]{(abc)^2}$,

$$\text{then } p\geq\sqrt{3}, r\leq\frac{1}{3\sqrt{3}}, \text{ and}$$

$$3\lambda^2-2\sqrt{3}\lambda+1+\lambda(2-\sqrt{3}\lambda)p-3\sqrt{3}r\geq 3\lambda^2-2\sqrt{3}\lambda+1+\lambda(2-\sqrt{3}\lambda).\sqrt{3}-1=0,$$

which completes the proof. Equality holds iff $a=b=c=\frac{\sqrt{3}}{3}$.

1347. If $x,y,z>0$ then:

$$2\sum_{cyc}(x+y)^4\geq 96xyz(x+y+z)+\sum_{cyc}(y-x)(x+y+2z)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das – India

NOTE: $\sum(y-x)(x+y+2z)=\sum(y-x)\{(x+y+z)+z\}$

$$=(x+y+z)\sum(y-x)+\sum z(y-x)=(x+y+z)(y-x+z-y+x-z)$$

$$+(xy+yz+zx-xy+yz+zx)=0$$

Now the problem is

$$2\sum(x+y)^4 \ge 32xyz(x+y+z)$$

Now

$$\sum(x+y)^4 \overset{CBS}{\ge} \frac{(x+y+y+z+z+x)^4}{27} = \frac{1}{27}\cdot 16(x+y+z)^4$$

$$= \frac{16}{27}(x+y+z)^3(x+y+z) \overset{AM-GM}{\ge} \frac{16}{27}\left[3(xyz)^{\frac{1}{3}}\right]^3(x+y+z)$$

$$= 16(x+y+z)xyz$$

$$\therefore 2\sum(x+y)^4 \ge 2\times 16(x+y+z)xyz = 32(xyz)(x+y+z)$$

Solution 2 by Tapas Das – India

Note: $\sum(y-x)(x+y+2z) = \sum(y-x)(x+y+z) + \sum(y-x)z = 0$

Now the problem is

$$6\sum(x+y)^4 \ge 96xyz(x+y+z)$$

Now $\sum(x+y)^4 \overset{CBS}{\ge} \frac{[2(\sum x)]^4}{27} = \frac{16}{27}(\sum x)^3(\sum x)$

$$\overset{AM-GM}{\ge} \frac{16}{27}\cdot 27(xyz)\left(\sum x\right) = 16xyz\left(\sum x\right)$$

$$\therefore 6\sum(x+y)^4 \ge 16\times 6xyz(x+y+z) = 96xyz(x+y+z)$$

1348. $x,y,z\in\mathbb{R}^+$, $xyz \le xy+xz+yz$. **Prove that:**

$$\frac{x}{yz+y+z}+\frac{y}{xz+x+z}+\frac{z}{xy+y+x} \ge \frac{3}{5}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Lazaros Zachariadis-Greece

$$LHS = \frac{x}{yz+y+z}+\frac{y}{xz+x+z}+\frac{z}{xy+x+y} =$$

$$= \frac{x^2}{xyz+yx+zx}+\frac{y^2}{xyz+xy+yz}+\frac{z^2}{xyz+xz+yz}$$

$$\overset{Bergstrom}{\geq}\frac{(x+y+z)^2}{3xyz+2(xy+yz+zx)}\geq\frac{(x+y+z)^2}{3(xy+yz+zx)+2(xy+yz+zx)}$$

$$\geq\frac{3(xy+yz+zx)}{5(xy+yz+zx)}=\frac{3}{5}=RHS$$

1349. Let $a,b,c>0: abc=1$. *Prove that*

$$\sqrt{\frac{a^3+3}{a+3}}+\sqrt{\frac{b^3+3}{b+3}}+\sqrt{\frac{c^3+3}{c+3}}\geq\frac{a+b+c}{3}+2$$

Proposed by Nguyen Thai An-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c,\ \ q:=ab+bc+ca,\ \ r:=abc=1$.

By CBS inequality, we have

$$\sum_{cyc}\sqrt{\frac{a^3+3}{a+3}}\geq\sum_{cyc}\frac{a^2+3}{a+3}=\sum_{cyc}\left(a-3+\frac{12}{a+3}\right)=p-9+\frac{12(q+6p+27)}{r+3q+9p+27}$$

$$=p-9+4\left(1+\frac{9p+53}{3q+9p+28}\right)\overset{3q\leq p^2}{\geq}p-5+\frac{4(9p+53)}{p^2+9p+28}$$

$$=\frac{p}{3}+2+\frac{(p-3)(2p^2+3p-16)}{p^2+9p+28}\overset{p\geq 3}{\geq}\frac{p}{3}+2=\frac{a+b+c}{3}+2.$$

as desired. Equality holds iff $a=b=c=1$.

1350. Let $a,b,c\geq 0:\ a^2+b^2+c^2+2abc=1$. Prove that

$$abc+1\geq a\sqrt{b^2+b^2c^2+c^2}+b\sqrt{c^2+c^2a^2+a^2}+c\sqrt{a^2+a^2b^2+b^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

If two of the numebrs a,b,c are zero, the inequality is true. Assume now that $ab+bc+ca\neq 0$. The given condition can be rewritten as follows

$$\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{c}{c+ab}=2.$$

By AM − GM inequality, we have

$$2a\sqrt{b^2+b^2c^2+c^2}\leq(a^2+abc)+\frac{a(b^2+b^2c^2+c^2)}{a+bc}=2abc+\frac{a(a^2+b^2+c^2)}{a+bc}.$$

Therefore

$$\sum_{cyc} a\sqrt{b^2+b^2c^2+c^2} \le \sum_{cyc}\left(abc+\frac{a^2+b^2+c^2}{2}.\frac{a}{a+bc}\right) = 3abc+\frac{a^2+b^2+c^2}{2}.2$$
$$= abc+1.$$

Equality holds iff $a=b=c=\frac{1}{2}$ or $a=b=\frac{\sqrt{2}}{2}, c=0$ and permutations.

1351. Let $a,b,c>0: abc=1$. Prove that

$$\frac{1}{\sqrt{a+3}}+\frac{1}{\sqrt{b+3}}+\frac{1}{\sqrt{c+3}} \le \frac{a+b+c}{2}$$

Proposed by Nguyen Thai An-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sum_{cyc}\frac{1}{\sqrt{a+3}} \le \sum_{cyc}\frac{1}{\sqrt{4\sqrt[4]{a}}} = \frac{1}{2}\sum_{cyc}\sqrt[8]{bc} \le \frac{1}{2}\sum_{cyc}\frac{b+c+6}{8} = \frac{a+b+c}{8}+\frac{9}{8} \le \frac{a+b+c}{2},$$

because $a+b+c \ge 3\sqrt[3]{abc}=3$. Equality holds iff $a=b=c=1$.

1352. Let $a,b,c \ge 0: ab+bc+ca>0$. Prove that

$$\frac{a}{2a+\sqrt{bc}}+\frac{b}{2b+\sqrt{ca}}+\frac{c}{2c+\sqrt{ab}}+\frac{a^2+b^2+c^2}{3(ab+bc+ca)} \ge \frac{4}{3}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$LHS = \frac{a^2}{2a^2+a\sqrt{bc}}+\frac{b^2}{2b^2+b\sqrt{ca}}+\frac{c^2}{2c^2+c\sqrt{ab}}+\frac{(a+b+c)^2}{3(ab+bc+ca)}-\frac{2}{3}$$
$$\ge \frac{[a+b+c+(a+b+c)]^2}{\left(2a^2+a\sqrt{bc}\right)+\left(2b^2+b\sqrt{ca}\right)+\left(2c^2+c\sqrt{ab}\right)+3(ab+bc+ca)}-\frac{2}{3}$$
$$= \frac{4(a+b+c)^2}{2(a+b+c)^2-\left[(ab+bc+ca)-\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\right]}-\frac{2}{3} \ge 2-\frac{2}{3}=\frac{4}{3},$$

the last line is true by the AM − GM inequality,

$$\therefore ab+bc+ca \ge \sqrt{ab}.\sqrt{bc}+\sqrt{bc}.\sqrt{ca}+\sqrt{ca}.\sqrt{ab} = \sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

Equality holds iff $a=b=c>0$.

1353. Let $a,b,c\ge 0: ab+bc+ca+abc=4$. Prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\ge\frac{2(a+b+c-3)+3\sqrt{abc}}{\sqrt{abc}+2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{2(a+b+c-3)+3\sqrt{abc}}{\sqrt{abc}+2}=\frac{4\left(6-3\sqrt{abc}\right)\left[2(a+b+c-3)+3\sqrt{abc}\right]}{12(4-abc)}$$

$$\overset{AM-GM}{\le}\frac{\left[\left(6-3\sqrt{abc}\right)+2(a+b+c-3)+3\sqrt{abc}\right]^2}{12(ab+bc+ca)}=$$

$$=\frac{(a+b+c)^2}{3(ab+bc+ca)}\le\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Equality holds iff $a=b=c=1$.

1354. Let $a,b,c>0: ab+bc+ca=3$. Prove that

$$\sqrt{\frac{a}{a^3+3}}+\sqrt{\frac{b}{b^3+3}}+\sqrt{\frac{c}{c^3+3}}\le\frac{3}{2}$$

Proposed by Nguyen Thai An-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $(a+b+c)^2\ge 3(ab+bc+ca)$ & $ab+bc+ca\ge 3\sqrt[3]{(abc)^2}$ then $a+b+c\ge 3,\ \ abc\le 1$.

$$\Rightarrow\sum_{cyc}\sqrt{\frac{a}{a^3+3}}=\sum_{cyc}\frac{2\sqrt{4a(a+3)}}{4\sqrt{(a^3+3)(a+3)}}\overset{AM-GM\ \&\ CBS}{\le}$$

$$\le\sum_{cyc}\frac{4a+(a+3)}{4(a^2+3)}=\sum_{cyc}\frac{5a+3}{4(a+b)(a+c)}$$

$$=\frac{10(ab+bc+ca)+6(a+b+c)}{4(a+b)(b+c)(a+c)}=\frac{15+3(a+b+c)}{2[3(a+b+c)-abc]}\le$$

$$\le\frac{3(5+a+b+c)}{2[3(a+b+c)-1]}\le\frac{3}{2}.$$

Equality holds iff $a=b=c=1$.

1355. If $a,b,c>0$ and $\lambda\ge 0$ then

$$\sum_{cyc}\frac{b^2}{(\lambda+1)a^2+(2\lambda+3)ab+\lambda b^2}\ge\frac{3}{4(\lambda+1)}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}=1$ then $\exists x,y,z>0$ such that $\frac{a}{b}=\frac{yz}{x^2},\ \frac{b}{c}=\frac{zx}{y^2},\ \frac{c}{a}=\frac{xy}{z^2}$.

By CBS inequality, we have

$$\sum_{cyc}\frac{b^2}{(\lambda+1)a^2+(2\lambda+3)ab+\lambda b^2}=\sum_{cyc}\frac{x^4}{(\lambda+1)(yz)^2+(2\lambda+3)yz.x^2+\lambda x^4}$$

$$\ge\frac{(x^2+y^2+z^2)^2}{(\lambda+1)\sum_{cyc}(yz)^2+(2\lambda+3)\sum_{cyc}xy.zx+\lambda\sum_{cyc}x^4}$$

$$\ge\frac{(x^2+y^2+z^2)^2}{(\lambda+1)\sum_{cyc}(yz)^2+(2\lambda+3)\sum_{cyc}(yz)^2+\lambda\sum_{cyc}x^4}$$

$$=\frac{3(x^2+y^2+z^2)^2}{(\lambda+4).3\sum_{cyc}(yz)^2+3\lambda\left(\sum_{cyc}x^2\right)^2}$$

$$\ge\frac{3(x^2+y^2+z^2)^2}{(\lambda+4)\left(\sum_{cyc}x^2\right)^2+3\lambda\left(\sum_{cyc}x^2\right)^2}=\frac{3}{4(\lambda+1)}.$$

Equality holds iff $a=b=c$.

1356. If $a,b,c>0$ *such that* : $a+b+c=1$ *and* $1\le\lambda\le 3$, then :

$$\sum_{cyc}\frac{1}{\lambda a^2+b+c}\le\frac{27}{\lambda+6}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, x\in(0,1)\ and\ \forall\,\lambda\in[1,3],\frac{1}{\lambda(1-x)^2+x}\overset{?}{\le}\frac{9}{\lambda+6}-\frac{9(3-2\lambda)(3x-2)}{(\lambda+6)^2}$$

$$=\frac{9(\lambda+6)-9(3-2\lambda)(3x-2)}{(\lambda+6)^2}$$

$$\Leftrightarrow\lambda^2\left(54x^3-135x^2+108x-28\right)-\lambda\left(81x^3-324x^2+324x-96\right)$$

$$-\left(81x^2-108x+36\right)\overset{?}{\ge}0$$

$$\Leftrightarrow\lambda^2(6x-7)(3x-2)^2-3\lambda(3x-8)(3x-2)^2-9(3x-2)^2\overset{?}{\ge}0$$

$$\Leftrightarrow\lambda^2(6x-7)-3\lambda(3x-8)-9\underset{(*)}{\overset{?}{\ge}}0$$

$\boxed{\text{Case 1}}$ $2\lambda<3$ and we have : $\lambda^2(6x-7)-3\lambda(3x-8)-9$

$$= x(9\lambda^2-9\lambda)-3x\lambda^2-7\lambda^2+24\lambda-9 \overset{\lambda\ge 1\ and\ x<1}{\ge}$$
$$-10\lambda^2+24\lambda-9=(3-2\lambda)(5(\lambda-1)+2)+3\lambda$$
$$\overset{1\le\lambda<\frac{3}{2}}{\ge} 3\lambda>0 \Rightarrow (*) \text{ is true}$$
$$\boxed{\text{Case 2}}\ 2\lambda\ge 3 \text{ and we have}: \lambda^2(6x-7)-3\lambda(3x-8)-9$$
$$=\lambda(3x(2\lambda-3)+24-7\lambda)-9 \overset{2\lambda\ge 3}{\ge} 24\lambda-7\lambda^2-9=(3-\lambda)(7(\lambda-1)+4)\ge 0$$
$$\left(\because 1<\frac{3}{2}\le\lambda\le 3\right)\Rightarrow (*) \text{ is true} \therefore \text{combining cases 1 and 2}, (*)\Rightarrow$$
$$\frac{1}{\lambda(1-x)^2+x}\le\frac{9}{\lambda+6}-\frac{9(3-2\lambda)(3x-2)}{(\lambda+6)^2} \text{ is true } \forall\, x\in(0,1) \text{ and } \forall\,\lambda\in[1,3]$$
$$\to (1)\ \therefore \sum_{cyc}\frac{1}{\lambda a^2+b+c}\overset{a+b+c=1}{=}\sum_{cyc}\frac{1}{\lambda\left(1-(b+c)\right)^2+b+c}$$
$$=\sum_{cyc}\frac{1}{\lambda(1-x)^2+x}\ \left(x=b+c, y=c+a, z=a+b \text{ and } x,y,z\in(0,1)\right)$$
$$\overset{\text{via (1) and analogs}}{\le}\sum_{cyc}\left(\frac{9}{\lambda+6}-\frac{9(3-2\lambda)(3x-2)}{(\lambda+6)^2}\right)$$
$$=\frac{27}{\lambda+6}-\frac{9(3-2\lambda)}{(\lambda+6)^2}*\left(3\sum_{cyc}x-6\right)=\frac{27}{\lambda+6}-\frac{9(3-2\lambda)}{(\lambda+6)^2}*\left(3\sum_{cyc}(b+c)-6\right)$$
$$\overset{a+b+c=1}{=}\frac{27}{\lambda+6}-\frac{9(3-2\lambda)}{(\lambda+6)^2}*(6-6)=\frac{27}{\lambda+6}$$
$$\therefore \sum_{cyc}\frac{1}{\lambda a^2+b+c}\le\frac{27}{\lambda+6}\ \forall\, a,b,c>0 \mid a+b+c=1 \text{ and } 1\le\lambda\le 3,$$
$$''='' \text{ iff } x=y=z=\frac{2}{3}\Rightarrow \text{iff } a=b=c=\frac{1}{3}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove a lemma that for all $a\in(0,1)$ and $1\le\lambda\le 3$,

$$\frac{1}{\lambda a^2+1-a}\le\frac{27[\lambda+1-(2\lambda-3)a]}{(\lambda+6)^2} \qquad (1)$$

We have

$$27[\lambda+1-(2\lambda-3)a](\lambda a^2+1-a)-(\lambda+6)^2$$
$$=-\lambda^2+15\lambda-9-27(3\lambda-2)a+27(\lambda^2+3\lambda-3)a^2-27\lambda(2\lambda-3)a^3$$
$$=(3a-1)^2[-\lambda^2+15\lambda-9-3\lambda(2\lambda-3)a]\ge 0,$$

because $-\lambda^2+15\lambda-9-3\lambda(2\lambda-3)a$
$$=[(9-\lambda)(\lambda-1)+5\lambda](1-a)+(3-\lambda)(7\lambda-3)a\ge 0.$$

completing the proof of (1). Equality holds iff $a=1$.

Returning to the proposed inequality, by using (1), we have

$$\sum_{cyc}\frac{1}{\lambda a^2+b+c}=\sum_{cyc}\frac{1}{\lambda a^2+1-a}\le\sum_{cyc}\frac{27[\lambda+1-(2\lambda-3)a]}{(\lambda+6)^2}=\frac{27}{\lambda+6}.$$

Equality holds iff $a=b=c=1$.

1357. Let $a, b, c, d \geq 0$ such that $a^2 + b^2 + c^2 + d^2 = 2023$.

Prove that : $\dfrac{1}{a^4+2} + \dfrac{1}{b^4+2} + \dfrac{1}{c^4+2} + \dfrac{1}{d^4+2} \geq \dfrac{64}{4092561}$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

We first prove that : $\dfrac{1}{x^2+2} \geq \dfrac{1}{\left(\frac{2023}{4}\right)^2+2} - \left(x - \dfrac{2023}{4}\right) \cdot \dfrac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2}$

$$\forall\, x \geq 0 \Leftrightarrow \frac{1}{x^2+2} - \frac{16}{4092561} + \frac{64736(4x-2023)}{4092561^2} \geq 0$$

$$\Leftrightarrow \frac{2023^2 - 16x^2}{4092561(x^2+2)} - \frac{64736(2023-4x)}{4092561^2} \geq 0$$

$$\Leftrightarrow \frac{2023-4x}{4092561}\left(\frac{2023+4x}{x^2+2} - \frac{32.2023}{2023^2+32}\right) \geq 0$$

$$\Leftrightarrow \left(\frac{m-4x}{4092561}\right) \cdot \frac{(m+4x)(m^2+32) - 32m(x^2+2)}{(m^2+32)(x^2+2)} \geq 0 \ (m = 2023)$$

$$\Leftrightarrow \left(\frac{m-4x}{4092561}\right) \cdot \frac{m^3 + 4m^2x - 32mx^2 - 32m + 128x}{(m^2+32)(x^2+2)} \geq 0$$

$$\Leftrightarrow \left(\frac{m-4x}{4092561}\right) \cdot \frac{(m-4x)(m^2-32+8mx)}{(m^2+32)(x^2+2)} \geq 0$$

$$\Leftrightarrow \frac{(2023-4x)^2(4092497 + 8.2023x)}{4092561(2023^2+32)(x^2+2)} \geq 0 \to \text{true} \because x \geq 0$$

$$\therefore \frac{1}{x^2+2} \geq \frac{1}{\left(\frac{2023}{4}\right)^2+2} - \left(x - \frac{2023}{4}\right) \frac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2} \overset{x \equiv a^2, b^2, c^2, d^2}{\Rightarrow}$$

$$\frac{1}{a^4+2} \geq \frac{1}{\left(\frac{2023}{4}\right)^2+2} - \left(a^2 - \frac{2023}{4}\right) \cdot \frac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2} \text{ and analogs}$$

$$\overset{\text{summation}}{\Rightarrow} \frac{1}{a^4+2} + \frac{1}{b^4+2} + \frac{1}{c^4+2} + \frac{1}{d^4+2} \geq$$

$$\frac{4}{\left(\frac{2023}{4}\right)^2+2} - \frac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2} \cdot \left(\begin{matrix} a^2+b^2+ \\ c^2+d^2-2023 \end{matrix}\right) \overset{a^2+b^2+c^2+d^2=2023}{=} \frac{4}{\left(\frac{2023}{4}\right)^2+2}$$

$$=\frac{64}{4092561}\therefore\frac{1}{a^4+2}+\frac{1}{b^4+2}+\frac{1}{c^4+2}+\frac{1}{d^4+2}\ge\frac{64}{4092561}$$

$$\forall\, a,b,c,d\ge 0\,\middle|\,a^2+b^2+c^2+d^2=2023,''='' \text{ iff } a=b=c=d=\frac{\sqrt{2023}}{2}\text{ (QED)}$$

1358. If $x,y,z>0$, *then* :

$$\sum_{cyc}x^8z^4\cdot\sum_{cyc}\frac{1}{(xy^2+yz^2)^4}\ge\frac{9}{16}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}x^8z^4\cdot\sum_{cyc}\frac{1}{(xy^2+yz^2)^4}=\sum_{cyc}x^4y^8\cdot\sum_{cyc}\frac{1^5}{(xy^2+yz^2)^4}\overset{Radon}{\ge}$$

$$\sum_{cyc}x^4y^8\cdot\frac{3^5}{\left(\sum_{cyc}xy^2+\sum_{cyc}yz^2\right)^4}=\frac{243\left(\sum_{cyc}(xy^2)^4\right)}{16\left(\sum_{cyc}xy^2\right)^4}\overset{Holder}{\ge}\frac{243.\frac{\left(\sum_{cyc}xy^2\right)^4}{27}}{16\left(\sum_{cyc}xy^2\right)^4}=\frac{9}{16},$$

$$\therefore\sum_{cyc}x^8z^4\cdot\sum_{cyc}\frac{1}{(xy^2+yz^2)^4}\ge\frac{9}{16}\ \forall\, x,y,z>0,''='' \text{ iff } x=y=z\text{ (QED)}$$

1359. If $x,y,z>0$, *then* :

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\ge 4$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b$
$\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(i)}{=}s$

$$\Rightarrow x=s-a, y=s-b, z=s-c$$

Via such substitutions, $xyz=(s-a)(s-b)(s-c)\Rightarrow xyz\overset{(ii)}{=}r^2s$ and

$$\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)=4Rr+r^2\Rightarrow\sum_{cyc}xy\overset{(1)}{=}4Rr+r^2$$

$$\Rightarrow\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{via\ (i),(1)}{=}s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} x^2 \overset{(2)}{=} s^2 - 8Rr - 2r^2 \therefore \frac{xy+yz+zx}{x^2+y^2+z^2} + \frac{(x+y+z)^3}{9xyz} \overset{\text{via (i),(ii),(1),(2)}}{=}$$

$$\frac{4Rr+r^2}{s^2-8Rr-2r^2} + \frac{s^3}{9r^2s} = \frac{9r^2(4Rr+r^2)+s^2(s^2-8Rr-2r^2)}{9r^2(s^2-8Rr-2r^2)} \overset{?}{\geq} 4$$

$$\Leftrightarrow s^4 - (8Rr+38r^2)s^2 + 81r^3(4R+r) \underbrace{\overset{?}{\geq}}_{(*)} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (8Rr-43r^2)s^2 + 81r^3(4R+r) \overset{?}{\geq} 0$$

$$\Leftrightarrow (8R-43r)s^2 + 81r^2(4R+r) \underbrace{\overset{?}{\geq}}_{(**)} 0$$

$\boxed{Case\ 1}$ $8R - 43r \geq 0$ *and then* : LHS of $(*) \geq 81r^2(4R+r) > 0$

$\Rightarrow (**)$ is true (strict inequality)

$\boxed{Case\ 2}$ $8R - 43r < 0$ *and then* : LHS of $(*) = -(43r-8R)s^2 + 81r^2(4R+r)$

$$\overset{\text{Gerretsen}}{\geq} -(43r-8R)(4R^2+4Rr+3r^2) + 81r^2(4R+r) \overset{?}{\geq} 0$$

$$\Leftrightarrow 8t^3 - 35t^2 + 44t - 12 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (8t-3)(t-2)^2 \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (**)$ is true $\therefore$ combining cases 1, 2, $(**) \Rightarrow (*)$ is true $\forall$ triangles of sides a, b, c

$$\Rightarrow \frac{xy+yz+zx}{x^2+y^2+z^2} + \frac{(x+y+z)^3}{9xyz} \geq 4\ \forall x,y,z > 0, ''='' \text{ iff } x = y = z \text{ (QED)}$$

1360. Let $a, b, c > 0$ and $abc = 1$. Prove that :

$$\frac{a}{a^2+(b+c)^2} + \frac{b}{b^2+(c+a)^2} + \frac{c}{c^2+(a+b)^2} \leq \frac{3}{5}$$

Proposed by Nguyen Thai An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a^2+(b+c)^2} + \frac{b}{b^2+(c+a)^2} + \frac{c}{c^2+(a+b)^2} = \sum_{cyc} \frac{a}{\lambda + 2bc} \left(\lambda = \sum_{cyc} a^2\right)$$

$$= \sum_{cyc} \frac{a(\lambda+2ca)(\lambda+2ab)}{(\lambda+2ab)(\lambda+2bc)(\lambda+2ca)}$$

$$= \frac{\lambda^2(\sum_{cyc} a) + 2\lambda(\sum_{cyc} a)(\sum_{cyc} ab) - 6\lambda abc + 4\lambda}{\lambda^3 + 2\lambda^2(\sum_{cyc} ab) + 4\lambda abc(\sum_{cyc} a) + 8(abc)^2}$$

$$= \frac{\lambda(\sum_{cyc} a)(\sum_{cyc} a^2 + 2\sum_{cyc} ab) - 2\lambda abc}{\lambda^2(\sum_{cyc} a^2 + 2\sum_{cyc} ab) + 4\lambda abc(\sum_{cyc} a) + 8(abc)^2}$$

$$= \frac{(\sum_{cyc} a^2)\left((\sum_{cyc} a)^3 - 2abc\right)}{(\sum_{cyc} a^2)^2(\sum_{cyc} a)^2 + 4abc(\sum_{cyc} a)(\sum_{cyc} a^2) + 8(abc)^2} \overset{(*)}{=}$$

$$\frac{a}{a^2+(b+c)^2} + \frac{b}{b^2+(c+a)^2} + \frac{c}{c^2+(a+b)^2}$$

$$\text{Now, } \sum_{cyc} ab \overset{A-G}{\geq} 3\sqrt[3]{(abc)^2} \overset{abc=1}{=} 3 \Rightarrow \frac{\sum_{cyc} ab}{3} \geq 1 \Rightarrow 1 \overset{(**)}{\leq} \sqrt{\frac{\sum_{cyc} ab}{3}}$$

$$\therefore \frac{a}{a^2+(b+c)^2} + \frac{b}{b^2+(c+a)^2} + \frac{c}{c^2+(a+b)^2} \overset{\text{via } (*) \text{ and } (**)}{\leq}$$

$$\sqrt{\frac{\sum_{cyc} ab}{3}} \cdot \frac{\left(\sum_{cyc} a^2\right)\left(\left(\sum_{cyc} a\right)^3 - 2abc\right)}{\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a\right)^2 + 4abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) + 8(abc)^2} \overset{?}{\leq} \frac{3}{5}$$

$$\Leftrightarrow \frac{\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)^2\left(\left(\sum_{cyc} a\right)^3 - 2abc\right)^2}{\left(\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a\right)^2 + 4abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) + 8(abc)^2\right)^2} \underset{(\bullet)}{\overset{?}{\leq}} \frac{27}{25}$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a >0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$

$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius

$$= s, R, r \text{ (say) yielding } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1)$$

$\Rightarrow a = s-x, b = s-y, c = s-z \therefore abc = r^2 s \to (2)$ and such substitutions $\Rightarrow$

$$\sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3) \text{ and}$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via } (1) \text{ and } (3)}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4) \therefore \text{via } (1), (2), (3) \text{ and } (4), (\bullet)$$

$$\Leftrightarrow \frac{(4Rr+r^2)(s^2-8Rr-2r^2)^2(s^3-2r^2s)^2}{(s^2(s^2-8Rr-2r^2)^2+4r^2s^2(s^2-8Rr-2r^2)+8r^4s^2)^2} \leq \frac{27}{25}$$

$$\Leftrightarrow 27s^{10} - (964Rr + 25r^2)s^8 + r^2(11968R^2 + 1200Rr + 416r^2)s^6$$
$$-r^3(61696R^3 + 11200R^2r + 8256Rr^2 + 600r^3)s^4$$
$$+r^4(110592R^4 + 25600R^3r + 39424R^2r^2 + 8000Rr^3 + 1232r^4)s^2$$
$$-r^7(25600R^3 + 19200R^2r + 4800Rr^2 + 400r^3) \underset{(\bullet\bullet)}{\overset{?}{\geq}} 0$$

$$\text{Now, via Gerretsen, } 27(s^2 - 16Rr + 5r^2)^5$$
$$+(1196Rr - 700r^2)(s^2 - 16Rr + 5r^2)^4$$
$$+r^2(19392R^2 - 24320Rr + 7666r^2)(s^2 - 16Rr + 5r^2)^3$$
$$+r^3(137984R^3 - 282880R^2r + 197112Rr^2 - 44340r^3) \underset{(\blacksquare)}{\overset{?}{\geq}} 0$$

$\therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : LHS of $(\bullet\bullet) \geq$ LHS of $(\blacksquare) \Leftrightarrow$

$$\boxed{(380928R^4 - 1200640R^3r + 1469120R^2r^2 - 756320Rr^3 + 135307r^4)s^2 \overset{(\bullet\bullet\bullet)}{\geq} r\begin{pmatrix}5963776R^5 - 20029440R^4r + 26759976R^3r^2 \\ -16704320R^2r^3 + 4785300Rr^4 - 502975r^5\end{pmatrix}}$$

$\because 380928R^4 - 1200640R^3r + 1469120R^2r^2 - 756320Rr^3 + 135307r^4$

$$= (R-2r)\begin{pmatrix}161536R^3+219392R^2(R-2r)+\\591552Rr^2+426784r^3\end{pmatrix}+988875r^4 \overset{\text{Euler}}{\geq} 988875r^3>0$$

$$\therefore \text{LHS of } (\bullet\bullet\bullet) \overset{\text{Rouche}}{\geq} \begin{pmatrix}380928R^4-1200640R^3r+\\1469120R^2r^2-756320Rr^3+135307r^4\end{pmatrix}\begin{pmatrix}2R^2+10Rr\\-r^2-2(R-2r)\sqrt{R^2-2Rr}\end{pmatrix}$$

$$\overset{?}{\geq} \text{RHS of } (\bullet\bullet\bullet) \Leftrightarrow 761856R^6-4555776R^5r+10580352R^4r^2$$
$$-12380736R^3r^3+7942614R^2r^4-2675910Rr^5+367668r^6$$
$$\overset{?}{\geq} 2(R-2r)\begin{pmatrix}380928R^4-1200640R^3r+1469120R^2r^2\\-756320Rr^3+135307r^4\end{pmatrix}\sqrt{R^2-2Rr}$$

$$\Leftrightarrow \boxed{\begin{matrix}(R-2r)\begin{pmatrix}761856R^5-3032064R^4r+4516224R^3r^2\\-3348288R^2r^3+1246038Rr^4-183834r^5\end{pmatrix}\overset{?}{\geq}\\2(R-2r)\begin{pmatrix}380928R^4-1200640R^3r+1469120R^2r^2\\-756320Rr^3+135307r^4\end{pmatrix}\sqrt{R^2-2Rr}\end{matrix}} \Leftrightarrow$$

$$761856R^5-3032064R^4r+4516224R^3r^2-3348288R^2r^3+1246038Rr^4$$
$$-183834r^5 \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\geq}} 2\begin{pmatrix}380928R^4-1200640R^3r+1469120R^2r^2\\-756320Rr^3+135307r^4\end{pmatrix}\sqrt{R^2-2Rr}$$

$$\left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)$$

$$\because 761856R^5-3032064R^4r+4516224R^3r^2$$
$$-3348288R^2r^3+1246038Rr^4-183834r^5$$
$$= (R-2r)\left((R-2r)\begin{pmatrix}761856R^3+15360R^2r+\\1530240Rr^2+2711232r^3\end{pmatrix}+5970006r^4\right)+911250r^5$$
$$\overset{\text{Euler}}{\geq} 911250r^5>0 \therefore (\bullet\bullet\bullet\bullet)$$

$$\Leftrightarrow \begin{pmatrix}761856R^5-3032064R^4r+4516224R^3r^2\\-3348288R^2r^3+1246038Rr^4-183834r^5\end{pmatrix}^2 \geq$$
$$4(R^2-2Rr)\begin{pmatrix}380928R^4-1200640R^3r+\\1469120R^2r^2-756320Rr^3+135307r^4\end{pmatrix}^2$$

$$\Leftrightarrow 49928994816t^9-371514671104t^8+1103367241728t^7$$
$$-1635652599808t^6+1182357585920t^5-168754001920t^4$$
$$-357325258240t^3+268267717248t^2$$
$$-77916106348t+8448734889 \geq 0 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \boxed{\boxed{(t-2)\Big((t-2)\big((t-2).P+1887506976384\big)+764336452500\Big)+207594140625\geq 0}}$$

$$\begin{pmatrix}\text{where } P=13958643712t^6+35970351104(t-2)t^5+72575090688t^4\\+62518329344t^3+111040856064t^2+327871907840t+777562550784\end{pmatrix}$$

$$\overset{\text{Euler}}{\geq} 207594140625>0 \Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\Rightarrow \frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(c+a)^2}+\frac{c}{c^2+(a+b)^2}\leq\frac{3}{5}$$

$$\forall a,b,c>0 \,\big|\, abc=1,''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Nguyen Van Canh-Vietnam

We have: $\frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(a+c)^2}+\frac{c}{c^2+(a+b)^2}\leq\frac{3}{5}$;

$$\Leftrightarrow \frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(a+c)^2}+\frac{c}{c^2+(a+b)^2}\leq\frac{3}{5\sqrt[3]{abc}}; \ (*)$$

WLOG, we suppose that: $a^2+b^2+c^2=3$. **We have:**

$$(*)\Leftrightarrow \frac{a}{3+2bc}+\frac{b}{3+2ac}+\frac{c}{3+2ab}\le\frac{3}{5\sqrt[3]{abc}};$$

$$\Leftrightarrow\frac{12abc+6\big(ab(a+b)+bc(b+c)+ca(c+a)\big)+9(a+b+c)}{8(abc)^2+12abc(a+b+c)+18(ab+bc+ca)+27}\le\frac{3}{5\sqrt[3]{abc}};$$

$$\Leftrightarrow\frac{12r^3+6(pq-3r)+9p}{8r^6+12pr+18q+27}\le\frac{3}{5r};$$

$$\left(\begin{array}{c}\therefore \text{ where}: p=a+b+c\le 3; q=ab+bc+ca; 0<r=abc\le 1;\ p^2-2q=3\\ \Rightarrow q=\dfrac{p^2-3}{2}\end{array}\right)$$

$$\Leftrightarrow\frac{3\left(p^3+4r^3-6r\right)}{9p^2+12pr+8r^6}\le\frac{3}{5r}\Leftrightarrow 5r(p^3+4r^3-6r)\le 9p^2+12pr+8r^6;$$

$$\Leftrightarrow 5rp^3-9p^2-12rp-8r^6+20r^4-30r^2\le 0;$$

$$\therefore f(p)=5rp^3-9p^2-12rp-8r^6+20r^4-30r^2,(0<p\le 3)$$

$$\Rightarrow f'(p)=3\left(5rp^2-4r-6p\right)=0\overset{p>0}{\Leftrightarrow}p_0=\frac{3+\sqrt{20r^2+9}}{5r}$$

- **Case 1:** $\frac{3+\sqrt{20r^2+9}}{5r}\le 3\Leftrightarrow\frac{18}{41}\le r\le 1$. **We have:**

p	0	$\frac{3+\sqrt{20r^2+9}}{5r}$	3
$f'(p)$	$-$	0	$+$
$f(p)$	↘		↗

- **Case 2:** $\frac{3+\sqrt{20r^2+9}}{5r}>3\Leftrightarrow 0<r<\frac{18}{41}$.

p	0	3	$\frac{3+\sqrt{20r^2+9}}{5r}$
$f'(p)$	$-$	$-$	0
$f(p)$	↘		

From Case 1 & Case 2 we have: $f(p)\le\max\{f(0),f(3)\}$.

- $f(0)=-8r^6+20r^4-30r^2=-2r^2(4r^4-10r^2+15)=-2r^2\left(4\left(r^2-\frac{5}{4}\right)^2+\frac{35}{4}\right)<0$
- $f(3)=-8r^6+20r^4-30r^2+99r-81$
$=(1-r)(8r^5+8r^4-12r^3-12r^2+18r-81)$
$=(1-r)[(r^3+r^2)(8r^2-12)+18r-81]\le 0,(0<r\le 1).$

Therefore, $f(p)\le\max\{f(0),f(3)\}\le 0\Rightarrow(*)$ ***TRUE***. Proved. Equality$\Leftrightarrow q=p=3,r=1\Leftrightarrow a=b=c=1$.

1361. Let $a,b,c>0:a^2+b^2+c^2=a+b+c$. Prove that:

$$\frac{\sqrt{2a^2+bc}}{a^2+bc}+\frac{\sqrt{2b^2+ca}}{b^2+ca}+\frac{\sqrt{2c^2+ab}}{c^2+ab}\ge\frac{3\sqrt3}{2}$$

Proposed by Nguyen Thai An-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{2a^2+bc}=\sqrt{a^2+a^2+(\sqrt{bc})^2}\ge\sqrt{\frac{(a+a+\sqrt{bc})^2}{3}}\overset{G-H}{\ge}\frac{2a+\frac{2bc}{b+c}}{\sqrt3}$$

$$=\frac{2(\sum_{cyc}ab)}{\sqrt3(b+c)}\text{ and analogs}\Rightarrow\frac{\sqrt{2a^2+bc}}{a^2+bc}+\frac{\sqrt{2b^2+ca}}{b^2+ca}+\frac{\sqrt{2c^2+ab}}{c^2+ab}\ge$$

$$\frac{2(\sum_{cyc}ab)}{\sqrt3}\cdot\sum_{cyc}\frac{1}{(b+c)(a^2+bc)}\overset{\text{Bergstrom}}{\ge}\frac{2(\sum_{cyc}ab)}{\sqrt3}\cdot\frac{9}{2(\sum_{cyc}a^2b+\sum_{cyc}ab^2)}$$

$$=\frac{9(\sum_{cyc}ab)}{\sqrt3\left((\sum_{cyc}a)(\sum_{cyc}ab)-3abc\right)}\overset{?}{\ge}\frac{3\sqrt3}{2}\Leftrightarrow 2\sum_{cyc}ab\overset{?}{\ge}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc$$

$$\overset{\sum_{cyc}a^2=\sum_{cyc}a}{\Leftrightarrow}2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\overset{?}{\ge}\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)\left(\sum_{cyc}ab\right)$$

$$-3abc\left(\sum_{cyc}a\right)\Leftrightarrow\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\overset{?}{\ge}2\sum_{cyc}a^2b^2+abc\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow\sum_{cyc}a^3b+\sum_{cyc}ab^3\overset{?}{\ge}2\sum_{cyc}a^2b^2\Leftrightarrow\sum_{cyc}ab(a-b)^2\overset{?}{\ge}0\to\text{true}$$

$$\therefore\frac{\sqrt{2a^2+bc}}{a^2+bc}+\frac{\sqrt{2b^2+ca}}{b^2+ca}+\frac{\sqrt{2c^2+ab}}{c^2+ab}\ge\frac{3\sqrt3}{2}$$

$$\forall a,b,c>0\,|\,a^2+b^2+c^2=a+b+c,''='' \text{ iff } a=b=c=1\text{ (QED)}$$

1362. Let $a, b, c \geq 0$ such that : $ab + bc + ca = 1$. Prove that :

$$\frac{4(a^3+b^3+c^3)}{3} \geq a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2) \geq \frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2) \overset{1=ab+bc+ca}{=}$$

$$a\left(\sum_{cyc} ab - b^2\right)\left(\sum_{cyc} ab - c^2\right) + b\left(\sum_{cyc} ab - c^2\right)\left(\sum_{cyc} ab - a^2\right)$$

$$+c\left(\sum_{cyc} ab - a^2\right)\left(\sum_{cyc} ab - b^2\right) = 4abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2) \overset{(*)}{=} 4abc\left(\sum_{cyc} ab\right)$$

$$\therefore \frac{4(a^3+b^3+c^3)}{3} \geq a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2)$$

$$\overset{\text{via }(*)}{\Leftrightarrow} \frac{4(a^3+b^3+c^3)}{3} \geq 4abc\left(\sum_{cyc} ab\right) \overset{1=ab+bc+ca}{=} 4abc \Leftrightarrow a^3+b^3+c^3-3abc$$

$$\geq 0 \Leftrightarrow \frac{(a+b+c)}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \geq 0 \to \text{true} \because a, b, c \geq 0$$

$$\therefore \frac{4(a^3+b^3+c^3)}{3} \geq a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2)$$

$$\text{Also, } a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2) \geq \frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

$$\overset{\text{via }(*)}{\Leftrightarrow} 4abc\left(\sum_{cyc} ab\right)^4 \geq 108a^3b^3c^3 \Leftrightarrow \left(\sum_{cyc} ab\right)^3 \overset{(i)}{\geq} 27a^2b^2c^2$$

$$\text{Now, } (x+y+z)^3 - 27xyz = \sum_{cyc} x^3 + 3\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - xyz\right) - 27xyz$$

$$= 3xyz + \left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) - 3xyz + 3\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 27xyz$$

$$= \left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) + 3\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right)$$

$$=\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)+3\left(\begin{matrix}(y^2x+z^2x-2xyz)+(z^2y+x^2y-2xyz)+\\(x^2z+y^2z-2xyz)\end{matrix}\right)$$

$$=\frac{(\sum_{cyc}x)}{2}.\sum_{cyc}(x-y)^2+3\left(\sum_{cyc}x(y-z)^2\right)\geq 0\ \forall\ x,y,z\geq 0 \text{ and choosing}$$

$$x=ab, y=bc, z=ca, \text{we arrive at}: \left(\sum_{cyc}ab\right)^3\geq 27a^2b^2c^2\Rightarrow \text{(i) is true}$$

$$\therefore a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2)\geq\frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

$$\therefore \frac{4(a^3+b^3+c^3)}{3}\geq a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2)$$

$$\geq\frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}\ \forall\ a,b,c\geq 0\ \Big|\ ab+bc+ca=1,''='' \text{ iff } a=b=c=\frac{1}{\sqrt{3}}\ \text{(QED)}$$

1363. **Let $a,b,c\geq 0$ such that $a+b+c=3$. Prove that:**

$$28\min\left\{\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right),\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\right\}$$

$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, we shall prove}: 28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)-3\geq 27(ab+bc+ca)$$

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$ $(a=3)$

$$\text{and then}: \text{LHS}-\text{RHS}=28*3^{\frac{2023}{2024}}-3>3^3*3^{\frac{2023}{2024}}-3>0$$

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $(b+c=3)$

and then : LHS − RHS =

$$28.\left(\sqrt[2024]{\underbrace{b*b*b*\ldots b}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{c*c*c*\ldots c}_{2023\text{ terms}}*1}\right)-3-\frac{27.9bc}{(b+c)^2}$$

$$\overset{G-H}{\geq} 28\left(\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}\right)-3-\frac{243bc}{(b+c)^2}$$

$$=28.2024\left(\frac{b^2}{2023b+b^2}+\frac{c^2}{2023c+c^2}\right)-3-\frac{243bc}{(b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because b+c=3}}{\geq} \frac{28.2024(b+c)^2}{\frac{2023}{3}(b+c)^2+b^2+c^2} - 3 - \frac{243bc}{(b+c)^2}$$

$$= \frac{(84.2024-6069)(b^2+c^2+2bc)-9(b^2+c^2)}{2026(b^2+c^2)+2023.2bc} - \frac{243bc}{b^2+c^2+2bc}$$

$$= \frac{163938x+163947y}{2026x+2023y} - \frac{243y}{2(x+y)}\ (x=b^2+c^2; y=2bc)$$

$$= \frac{327876x^2+163452xy-163695y^2}{2(x+y)(2026x+2023y)}$$

$$= \frac{(327876x+491328y)(x-y)+327633y^2}{2(x+y)(2026x+2023y)} > 0 \because x=b^2+c^2 \geq 2bc = y$$

$$\text{and } x, y > 0 \text{ as } b, c > 0 \therefore 28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right) - 3 > 27(ab+bc+ca)$$

Case 3 $a, b, c > 0$ and then : LHS − RHS $\overset{\because a+b+c=3}{=}$

$$28.\left(\sqrt[2024]{\underbrace{a*a*a*\ldots a}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{b*b*b*\ldots b}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{c*c*c*\ldots c}_{2023\text{ terms}}*1}\right)-3$$

$$-\frac{27.9(ab+bc+ca)}{(a+b+c)^2} \overset{G-H}{\geq} 28\left(\frac{2024}{\frac{2023}{a}+1}+\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}\right)-3$$

$$-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$= 28.2024\left(\frac{a^2}{2023a+a^2}+\frac{b^2}{2023b+b^2}+\frac{c^2}{2023c+c^2}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because a+b+c=3}}{\geq} \frac{28.2024(a+b+c)^2}{\frac{2023}{3}(a+b+c)^2+a^2+b^2+c^2} - 3 - \frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$= \frac{84.2024(u+2v)}{2023(u+2v)+3u} - 3 - \frac{243v}{u+2v}\ (u=a^2+b^2+c^2, v=ab+bc+ca)$$

$$= \frac{6(27323u^2+27242uv-54565v^2)}{2026u+4046v} = \frac{6(u-v)(27323u+54565v)}{2026u+4046v} \geq 0$$

$$(\because a^2+b^2+c^2 \geq ab+bc+ca \Rightarrow u \geq v \text{ and } a,b,c>0 \Rightarrow u,v>0)$$

$$\therefore 28\left(\sum_{\text{cyc}} a^{\frac{2023}{2024}}\right) - 3 \geq 27\left(\sum_{\text{cyc}} ab\right) \text{ and combining all cases,}$$

$$28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right) \geq 27(ab+bc+ca)+3$$

$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}$$

$$\left(\because a,b,c \geq 0 \text{ such that } a+b+c=3 \overset{\text{via A-G}}{\Rightarrow} 0 \leq abc \leq 1 \Rightarrow 0 \leq (abc)^{\frac{2025}{2026}} \leq 1\right)$$

$$\therefore \boxed{28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right) \overset{(*)}{\geq} 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}}$$

Now, we *shall* prove : $28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)-3\geq 27(ab+bc+ca)$

Case 1 *Exactly 2 variables* $=0$ *and* WLOG we *may assume* $b=c=0$ $(a=3)$

$$\text{and then : LHS}-\text{RHS}=28*3^{\frac{2024}{2025}}-3>3^3*3^{\frac{2024}{2025}}-3>0$$

Case 2 *Exactly 1 variable* $=0$ *and* WLOG we *may assume* $a=0$ $(b+c=3)$

and then : LHS − RHS =

$$28.\left(\sqrt[2025]{\underbrace{b*b*b*\dots b}_{2024\text{ terms}}*1}+\sqrt[2025]{\underbrace{c*c*c*\dots c}_{2024\text{ terms}}*1}\right)-3-\frac{27.9bc}{(b+c)^2}$$

$$\overset{G-H}{\geq}28\left(\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}\right)-3-\frac{243bc}{(b+c)^2}$$

$$=28.2025\left(\frac{b^2}{2024b+b^2}+\frac{c^2}{2024c+c^2}\right)-3-\frac{243bc}{(b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ and\\ \because b+c=3}}{\geq}\frac{28.2025(b+c)^2}{\frac{2024}{3}(b+c)^2+b^2+c^2}-3-\frac{243bc}{(b+c)^2}$$

$$=\frac{(84.2025-6072)(b^2+c^2+2bc)-9(b^2+c^2)}{2027(b^2+c^2)+2024.2bc}-\frac{243bc}{b^2+c^2+2bc}$$

$$=\frac{164019x+164028y}{2027x+2024y}-\frac{243y}{2(x+y)}\ \left(x=b^2+c^2;y=2bc\right)$$

$$=\frac{328038x^2+163533xy-163776y^2}{2(x+y)(2027x+2024y)}$$

$$=\frac{(328038x+491571y)(x-y)+327795y^2}{2(x+y)(2027x+2024y)}>0\ \because x=b^2+c^2\geq 2bc=y$$

$$and\ x,y>0\text{ as }b,c>0\ \therefore 28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)-3>27(ab+bc+ca)$$

Case 3 $a,b,c>0$ *and then* : LHS − RHS $\overset{\because a+b+c=3}{=}-\frac{27.9(ab+bc+ca)}{(a+b+c)^2}-3$

$$+28.\left(\sqrt[2025]{\underbrace{a*a*a*\dots a}_{2024\text{ terms}}*1}+\sqrt[2025]{\underbrace{b*b*b*\dots b}_{2024\text{ terms}}*1}+\sqrt[2025]{\underbrace{c*c*c*\dots c}_{2024\text{ terms}}*1}\right)$$

$$\overset{G-H}{\geq}28\left(\frac{2025}{\frac{2024}{a}+1}+\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=28.2025\left(\frac{a^2}{2024a+a^2}+\frac{b^2}{2024b+b^2}+\frac{c^2}{2024c+c^2}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ and\\ \because a+b+c=3}}{\geq}\frac{28.2025(a+b+c)^2}{\frac{2024}{3}(a+b+c)^2+a^2+b^2+c^2}-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{84.2025(u+2v)}{2024(u+2v)+3u}-3-\frac{243v}{u+2v}\ \left(u=a^2+b^2+c^2,v=ab+bc+ca\right)$$

$$=\frac{3(54673u^2+54511uv-109184v^2)}{2027u+4048v}=\frac{3(u-v)(54673u+109184v)}{2027u+4048v}\geq 0$$

$$(\because a^2+b^2+c^2\geq ab+bc+ca\Rightarrow u\geq v \text{ and } a,b,c>0\Rightarrow u,v>0)$$

$$\therefore 28\left(\sum_{cyc} a^{\frac{2024}{2025}}\right)-3\geq 27\left(\sum_{cyc} ab\right) \text{ and combining } all \text{ cases,}$$

$$28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\geq 27(ab+bc+ca)+3$$

$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}$$

$$\left(\because a,b,c\geq 0 \text{ such that } a+b+c=3\overset{\text{via A-G}}{\Rightarrow} 0\leq abc\leq 1\Rightarrow 0\leq (abc)^{\frac{2025}{2026}}\leq 1\right)$$

$$\therefore \boxed{28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\overset{(**)}{\geq} 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}}$$

$$\therefore (*),(**)\Rightarrow 28\min\left\{\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right),\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\right\}$$

$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}\ \forall a,b,c\geq 0\,|\,a+b+c=3,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1364. If $a,b,c>0$, then :

$$4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3\geq 7$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc} a=\sum_{cyc} x=2s\Rightarrow \sum_{cyc} a=s\to(1)\Rightarrow a=s-x,$ $b=s-y, c=s-z\therefore abc=r^2s\to(2)$

$$\therefore 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3\overset{\text{via (1) and (2)}}{=}4\sum_{cyc}\frac{s-x}{x}$$

$$+\frac{r^2s}{s^3}\left(\sum_{cyc}\frac{x}{y}\right)^3\overset{A-G}{\geq}\frac{4s(s^2+4Rr+r^2)}{4Rrs}-12+\frac{9r^2}{s^2}\left(\frac{1}{xyz}\sum_{cyc}\frac{x^2z^2}{z}\right)$$

$$\overset{\text{Bergstrom}}{\geq}\frac{s^2+4Rr+r^2}{Rr}-12+\frac{9r^2}{s^2}\left(\frac{1}{4Rrs}\cdot\frac{(s^2+4Rr+r^2)^2}{2s}\right)\overset{?}{\geq}7$$

$$\Leftrightarrow \frac{s^2+4Rr+r^2}{Rr}+\frac{9r(s^2+4Rr+r^2)^2}{8Rs^4}\overset{?}{\geq}19$$

$$\Leftrightarrow \boxed{8s^6-(120Rr+17r^2)s^4+r^2s^2(72R^2+18r^2)+9r^4(4R+r)^2 \overset{(\bullet)}{\geq} 0} \; and$$

$$\because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (\bullet), \text{it suffices to prove :}$$

$$\text{LHS of } (\bullet) \geq 8(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (264Rr-103r^2)s^4-r^2s^2(6144R^2-3912Rr+582r^2)$$

$$+r^3(32768R^3-30576R^2r+9672Rr^2-991r^3) \overset{(\bullet\bullet)}{\geq} 0 \; and$$

$$\because (264Rr-103r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (\bullet\bullet),$$

$$\text{it suffices to prove : LHS of } (\bullet\bullet) \geq (264Rr-103r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (288R^2-253Rr+56r^2)s^2 \overset{(\bullet\bullet\bullet)}{\geq} r(4352R^3-4754R^2r+1676Rr^2-198r^3)$$

$$\text{Now, } (288R^2-253Rr+56r^2)s^2 \overset{\text{Gerretsen}}{\geq} (288R^2-253Rr+56r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(4352R^3-4754R^2r+1676Rr^2-198r^3)$$

$$\Leftrightarrow 256t^3-734t^2+485t-82 \overset{?}{\geq} 0 \; \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(145t^2+111t(t-2)+41) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet)$$

$$\text{is true} \therefore 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3 \geq 7 \; \forall \, a,b,c>0,$$

$$''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Schur's inequality, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{(a^3+b^3+c^3+3abc)+a^2(b+c)+b^2(c+a)+c^2(a+b)}{(a+b)(b+c)(c+a)}$$

$$\geq \frac{2[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{(a+b)(b+c)(c+a)}$$

$$=2-\frac{4abc}{(a+b)(b+c)(c+a)}.$$

By AM − GM inequality, we have

$$(a+b)(b+c)(c+a) \geq \frac{8(a+b+c)(ab+bc+ca)}{9} \geq \frac{8\sqrt{3abc(a+b+c)^3}}{9}.$$

Using these inequalities and by AM − GM inequality, we obtain

$$4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 8-2\sqrt{\frac{27abc}{(a+b+c)^3}} \geq 8-\left(1+\frac{27abc}{(a+b+c)^3}\right)$$

$$=7-\frac{27abc}{(a+b+c)^3}.$$

Now by AM − GM inequality, we have

$$abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3 \geq abc\left(\frac{3}{a+b+c}\right)^3 = \frac{27abc}{(a+b+c)^3}.$$

Adding the last two inequalities yields the desired result.

Equality holds iff $a=b=c$.

1365. Let $a,b,c \geq 0$ such that $ab+bc+ca=1$. Prove that:

$$\frac{1}{\sqrt{a+bc}}+\frac{1}{\sqrt{b+ac}}+\frac{1}{\sqrt{c+ab}} \geq \frac{6}{\sqrt{2(a+b+c)+abc}}.$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Nguyen Van Canh-Vietnam

Let us denote $p=a+b+c \geq \sqrt{3(ab+bc+ca)}=\sqrt{3}$ and

$r=abc \geq \max\left\{0,\frac{p(4-p^2)}{4}\right\}$ (Schur's Inequality).

Using B.C.S Inequality we have:

$$\frac{1}{\sqrt{a+bc}}+\frac{1}{\sqrt{b+ac}}+\frac{1}{\sqrt{c+ab}} \geq \frac{9}{\sqrt{a+bc}+\sqrt{b+ac}+\sqrt{c+ab}} \geq \frac{9}{\sqrt{3(a+b+c+1)}}$$

$$=\frac{3\sqrt{3}}{\sqrt{a+b+c+1}}=\frac{3\sqrt{3}}{\sqrt{p+1}};$$

We need to prove that:

$$\frac{3\sqrt{3}}{\sqrt{p+1}} \geq \frac{6}{\sqrt{2p+r}} \quad (*) \Leftrightarrow 3(2p+r) \geq 4(p+1) \Leftrightarrow 2p+3r \geq 4;$$

- If $p \geq 2$ then $r \geq \max\left\{0,\frac{p(4-p^2)}{4}\right\}=0$ .We have:

  $2p+3r \geq 4+0=4$ (true) $\Rightarrow (*)$ true.
- If $\sqrt{3} \leq p \leq 2$ then $r \geq \max\left\{0,\frac{p(4-p^2)}{4}\right\}=\frac{p(4-p^2)}{4}$. We have:

  $2p+3r \geq 2p+3.\frac{p(4-p^2)}{4}=5p-\frac{3p^3}{4}$. We just prove that:

$$5p-\frac{3p^3}{4} \geq 4 \Leftrightarrow 3p^3-20p+16 \leq 0 \Leftrightarrow (p-2)\left(3p^2+6p-8\right) \leq 0;$$

$\Leftrightarrow (p-2)[(3p+9)(p-1)+1] \leq 0$ (true since $\sqrt{3} \leq p \leq 2$)

$\Rightarrow (*)$ true. Proved. Equality $\Leftrightarrow \begin{cases} a+b+c=2 \\ ab+bc+ca=1 \\ abc=0 \end{cases}$

$\Leftrightarrow a=b=1, c=0$ or $b=c=1=0$ or $a=c=1, b=0$.

1366. If $a,b,c>0.\ a^2+b^2+c^2=3$ and $\lambda \geq 0$, then :

$$\sum_{cyc}\frac{a+\lambda}{a+bc+c^2} \geq ab+bc+ca+\lambda-2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$3=\sum_{cyc}a^2\overset{A-G}{\geq}3\sqrt[2]{a^2b^2c^2}\Rightarrow abc\leq 1\rightarrow(1)\text{ and }3=\sum_{cyc}a^2\geq\frac{1}{3}\left(\sum_{cyc}a\right)^2$$

$$\Rightarrow\sum_{cyc}a\leq 3\rightarrow(2)$$

$$\text{Now, }\sum_{cyc}\frac{a}{a+bc+c^2}=\sum_{cyc}\frac{a^2}{a^2+abc+ac^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a^2+3abc+\sum_{cyc}a^2b}\overset{A-G}{\geq}$$

$$\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a^2+3abc+\sum_{cyc}a^3}=\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a^2+6abc+\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}$$

$$\overset{\text{via (1) and (2)}}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a^2+6+3\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}\overset{3=a^2+b^2+c^2}{=}$$

$$\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a^2+2\sum_{cyc}a^2+3\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}=\frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{6\sum_{cyc}a^2-3\sum_{cyc}ab}$$

$$\overset{?}{\geq}ab+bc+ca-2=\frac{3\sum_{cyc}ab}{\sum_{cyc}a^2}-2\Leftrightarrow\frac{x+2y}{6x-3y}+2\overset{?}{\geq}\frac{3y}{x}$$

$$\left(x=\sum_{cyc}a^2\text{ and }y=\sum_{cyc}ab\right)\Leftrightarrow\frac{13x-4y}{6x-3y}\overset{?}{\geq}\frac{3y}{x}\Leftrightarrow 13x^2-22xy+9y^2\overset{?}{\geq}0$$

$$\Leftrightarrow(13x-9y)(x-y)\overset{?}{\geq}0\rightarrow\text{true}\because x\geq y$$

$$\therefore\sum_{cyc}\frac{a}{a+bc+c^2}\geq ab+bc+ca-2\rightarrow(\text{i})$$

$$\text{Again, }\sum_{cyc}\frac{1}{a+bc+c^2}\overset{\text{Bergstrom}}{\geq}\frac{9}{\sum_{cyc}a+\sum_{cyc}ab+\sum_{cyc}a^2}\overset{\text{via (2)}}{\geq}\frac{9}{3+2\sum_{cyc}a^2}$$

$$\overset{a^2+b^2+c^2=3}{=}\frac{9}{3+6}\Rightarrow\sum_{cyc}\frac{1}{a+bc+c^2}\geq 1\text{ and }\because\lambda\geq 0\therefore\sum_{cyc}\frac{\lambda}{a+bc+c^2}\geq\lambda\rightarrow(\text{ii})$$

$$\therefore(\text{i})+(\text{ii})\Rightarrow\sum_{cyc}\frac{a+\lambda}{a+bc+c^2}\geq ab+bc+ca+\lambda-2$$

$$\forall a,b,c>0\,\middle|\,a^2+b^2+c^2=3\text{ and }\forall\lambda\geq 0,\text{"="}\text{ iff }a=b=c=1\text{ (QED)}$$

1367. Let $a,b,c,d\in\left[\frac{1}{2},1\right]$ and $a\geq b\geq c\geq d$. Prove that:

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\geq\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}+4$$

Proposed by Minh Vu-Vietnam

Solution by Nguyen Van Canh-Vietnam

$$\text{Let } f(a,b,c,d)=2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)-\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right)-4;$$

$$f'_a=\frac{2}{b}-\frac{1}{d}-\frac{2d}{a^2}+\frac{b}{a^2}=\frac{2d-b}{bc}-\frac{2d-b}{a^2}=(2d-b)\left(\frac{a^2-bc}{a^2bc}\right)\geq 0$$

$$\text{(since: } b,d\in\left[\frac{1}{2},1\right]\Rightarrow 2d-b\geq 1-b\geq 0 \text{ and } a\geq b\geq c\Rightarrow a^2-bc\geq 0)$$

$$\Rightarrow f(a,b,c,d)\geq f(b,b,c,d)=2\left(\frac{b}{c}+\frac{c}{d}+\frac{d}{b}\right)-\left(\frac{c}{b}+\frac{d}{c}+\frac{b}{d}\right)-3=g(b,c,d).$$

$$g'_b=\frac{2}{c}-\frac{2d}{b^2}+\frac{c}{b^2}-\frac{1}{d}=\frac{2d-c}{cd}-\frac{2d-c}{b^2}=(2d-c)\left(\frac{b^2-cd}{b^2cd}\right)\geq 0$$

$$\text{(since: } c,d\in\left[\frac{1}{2},1\right]\Rightarrow 2d-c\geq 1-c\geq 0 \text{ and } b\geq c\geq d\Rightarrow b^2-cd\geq 0)$$

$$\Rightarrow g(b,c,d)\geq g(c,c,d)=2\left(\frac{c}{d}+\frac{d}{c}\right)-\left(\frac{d}{c}+\frac{c}{d}\right)-2=\frac{d}{c}+\frac{c}{d}-2\overset{AM-GM}{\geq}2-2=0.$$

$$\Rightarrow f(a,b,c,d)\geq g(b,c,d)\geq 0.$$

$$\text{Proved. Equality} \Leftrightarrow a=b=c=d\in\left[\frac{1}{2};1\right].$$

1368. **If $a,b,c>0$, *then prove that* :**

$$\prod_{cyc}(a^2+ab+b^2)\leq\left(\frac{1}{2}(a-b)(b-c)(c-a)\right)^2+\sum_{cyc}(3ab)^2\left(\frac{a+b}{2}\right)^2$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(a^2+ab+b^2)\leq\left(\frac{1}{2}(a-b)(b-c)(c-a)\right)^2+\sum_{cyc}(3ab)^2\left(\frac{a+b}{2}\right)^2$$

$$\Leftrightarrow 9\sum_{cyc}a^2b^2(a+b)^2+\left((a-b)(b-c)(c-a)\right)^2\geq 4\prod_{cyc}(a^2+ab+b^2)$$

$$\Leftrightarrow\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+2\sum_{cyc}a^3b^3+3a^2b^2c^2\overset{(*)}{\geq}$$

$$abc\sum_{cyc}a^3+abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+6a^2b^2c^2$$

$$\text{Now, via Schur,}\sum_{cyc}a^3b^3+3a^2b^2c^2\geq abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\rightarrow(1)$$

$$\text{Again,}\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}a^4(b^2+c^2)\overset{A-G}{\geq}2\sum_{cyc}a^4bc=2abc\sum_{cyc}a^3$$

$$\Rightarrow\frac{1}{2}\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)\geq abc\sum_{cyc}a^3\rightarrow(2)$$

$$\text{Also, }\frac{1}{2}\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+\sum_{cyc}a^3b^3\overset{A-G}{\geq}\frac{1}{2}(3a^2b^2c^2+3a^2b^2c^2)+3a^2b^2c^2$$

$$\Rightarrow\frac{1}{2}\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+\sum_{cyc}a^3b^3\geq 6a^2b^2c^2\to(3)$$

$$\therefore (1)+(2)+(3)\Rightarrow(*)\text{ is true}$$

$$\therefore\prod_{cyc}(a^2+ab+b^2)\leq\left(\frac{1}{2}(a-b)(b-c)(c-a)\right)^2+\sum_{cyc}(3ab)^2\left(\frac{a+b}{2}\right)^2$$

$$\forall a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

Solution 2 by Nguyen Van Canh-Vietnam

WLOG, we assume that $a+b+c=1$. Let us denote $q=\sum ab$, $r=abc$.

We have:

$$4\prod(a^2+ab+b^2)\leq\left((a-b)(b-c)(c-a)\right)^2+9\sum(ab)^2(a+b)^2;$$

$$\Leftrightarrow 4\left[\sum a^2b^2(a^2+b^2)+\sum a^3b^3+abc\sum a^3+2abc\sum ab(a+b)+3a^2b^2c^2\right]$$
$$\leq\sum a^2b^2(a^2+b^2)+2abc\sum ab(a+b)-2\sum a^3b^3-6a^2b^2c^2$$
$$-2abc\sum a^3+9\left[\sum a^2b^2(a^2+b^2)+2\sum a^3b^3\right];$$

$$\Leftrightarrow\sum a^2b^2(a^2+b^2)+2\sum a^3b^3-abc\sum ab(a+b)-abc\sum a^3-3a^2b^2c^2\geq 0;$$

$$\Leftrightarrow\sum a^2b^2\sum a^2+2\sum a^3b^3-abc\sum ab(a+b)-abc\sum a^3-6a^2b^2c^2\geq 0;$$

$$\Leftrightarrow(q^2-2r)(1-2q)+2(q^3-3qr+3r^2)-r(q-3r)-r(1-3q+3r)-6r^2\geq 0;$$

$$\Leftrightarrow q^2\geq 3r.$$

Which is clearly true since: $(ab+bc+ca)^2\geq 3abc(a+b+c)\Rightarrow q^2\geq 3r$. Proved.

1369. If $a,b,c>0$, then prove that:

$$\sum_{cyc}\frac{4a}{3(2a+b+c)}\leq 1\Leftrightarrow\sum_{cyc}\frac{2a}{3(b+c)}\geq 1$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$, $y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$$\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1\Leftrightarrow\sum_{cyc}\frac{s-x}{(a+b)+(a+c)}\le\frac{3}{4}\Leftrightarrow\sum_{cyc}\frac{s-x}{(a+b)+(a+c)}\le\frac{3}{4}$$

$$\Leftrightarrow\sum_{cyc}\frac{s-x}{y+z}\le\frac{3}{4}\Leftrightarrow\sum_{cyc}\frac{2s-x-s}{2s-x}\le\frac{3}{4}\Leftrightarrow 3-\frac{3}{4}\le s\sum_{cyc}\frac{1}{y+z}$$

$$\Leftrightarrow\sum_{cyc}\frac{x^2+\sum_{cyc}xy}{\prod_{cyc}(y+z)}\ge\frac{9}{4s}\Leftrightarrow\frac{\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)+\sum_{cyc}xy}{\prod_{cyc}(y+z)}\ge\frac{9}{4s}$$

$$\Leftrightarrow\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}\ge\frac{9}{4s}\Leftrightarrow 10s^2+8Rr+2r^2\ge 9s^2+18Rr+9r^2$$

$$\Leftrightarrow s^2\ge 10Rr+7r^2\ \therefore\ \boxed{\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1\Leftrightarrow s^2\ge 10Rr+7r^2\to(1)}$$

$$\text{Also, }\sum_{cyc}\frac{2a}{3(b+c)}\ge 1\Leftrightarrow\sum_{cyc}\frac{s-x}{x}\ge\frac{3}{2}\Leftrightarrow s.\frac{\sum_{cyc}xy}{4Rrs}\ge\frac{9}{2}\Leftrightarrow\frac{s^2+4Rr+r^2}{4Rr}\ge\frac{9}{2}$$

$$\Leftrightarrow s^2\ge 14Rr-r^2\ \therefore\ \boxed{\sum_{cyc}\frac{2a}{3(b+c)}\ge 1\Leftrightarrow s^2\ge 14Rr-r^2\to(2)}$$

$$\text{If possible, let us assume }\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1\Rightarrow\sum_{cyc}\frac{2a}{3(b+c)}\le 1\text{, i.e.,}$$

$$\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1\text{ and }\sum_{cyc}\frac{2a}{3(b+c)}\le 1\text{ and then, via }(1),(2),$$

$$s^2\ge 10Rr+7r^2\text{ and }s^2\le 14Rr-r^2\text{ which is a contradiction}\because s^2-14Rr+r^2$$

$$=s^2-16Rr+5r^2+2r(R-2r)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\ge}0\Rightarrow s^2\ge 14Rr-r^2\ \therefore\text{ our asssumption}$$

$$\text{is incorrect}\ \therefore\ \boxed{\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1\Rightarrow\sum_{cyc}\frac{2a}{3(b+c)}\ge 1}$$

$$\text{Again, if possible, let us assume }\sum_{cyc}\frac{2a}{3(b+c)}\ge 1\Rightarrow\sum_{cyc}\frac{4a}{3(2a+b+c)}\ge 1\text{, i.e.,}$$

$$\sum_{cyc}\frac{2a}{3(b+c)}\ge 1\text{ and }\sum_{cyc}\frac{4a}{3(2a+b+c)}\ge 1\text{ and then, via }(1),(2),$$

$$s^2\ge 14Rr-r^2\text{ and }s^2\le 10Rr+7r^2\text{ which is a contradiction}\because s^2-10Rr-7r^2$$

$$=s^2-16Rr+5r^2+6r(R-2r)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\ge}0\Rightarrow s^2\ge 10Rr+7r^2\ \therefore\text{ our asssumption}$$

$$\text{is incorrect}\ \therefore\ \boxed{\sum_{cyc}\frac{2a}{3(b+c)}\ge 1\Rightarrow\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1}$$

$$\therefore\sum_{cyc}\frac{4a}{3(2a+b+c)}\le 1\Leftrightarrow\sum_{cyc}\frac{2a}{3(b+c)}\ge 1\text{ (QED)}$$

1370. If $x, y, z > 0$, then prove that:

$$\sum_{cyc}\frac{1}{2x+y+z}+\frac{16xyz}{\left(\sum_{cyc}x\right)\prod_{cyc}(2x+y+z)}\le\frac{5}{2\sum_{cyc}x}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0$,
$b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(\bullet)}{=}s\Rightarrow x=s-a, y=s-b, z=s-c$$

$$\therefore xyz\overset{(\bullet\bullet)}{=}r^2s$$

$$\text{Now, }\sum_{cyc}\frac{1}{2x+y+z}+\frac{16xyz}{\left(\sum_{cyc}x\right)\prod_{cyc}(2x+y+z)}\le\frac{5}{2\sum_{cyc}x}$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{(x+y)+(x+z)}+\frac{16xyz}{\left(\sum_{cyc}x\right)\prod_{cyc}((x+y)+(x+z))}\le\frac{5}{2\sum_{cyc}x}$$

$$\overset{\text{via }(\bullet)\text{ and }(\bullet\bullet)}{\Leftrightarrow}\sum_{cyc}\frac{1}{b+c}+\frac{16r^2s}{(s)\prod_{yc}(b+c)}\le\frac{5}{2s}$$

$$\Leftrightarrow\frac{1}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left(a^2+\sum_{cyc}ab\right)+\frac{16r^2}{2s(s^2+2Rr+r^2)}\le\frac{5}{2s}$$

$$\Leftrightarrow\frac{1}{2s(s^2+2Rr+r^2)}\cdot\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)+\frac{16r^2}{2s(s^2+2Rr+r^2)}\le\frac{5}{2s}$$

$$\Leftrightarrow\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}+\frac{16r^2}{2s(s^2+2Rr+r^2)}\le\frac{5}{2s}\Leftrightarrow\frac{5s^24Rr+17r^2}{2s(s^2+2Rr+r^2)}\le\frac{5}{2s}$$

$$\Leftrightarrow 6Rr\ge 12Rr\rightarrow\text{true via Euler}\therefore\sum_{cyc}\frac{1}{2x+y+z}+\frac{16xyz}{\left(\sum_{cyc}x\right)\prod_{cyc}(2x+y+z)}$$

$$\le\frac{5}{2\sum_{cyc}x}\ \forall x,y,z>0,''=''\text{ iff } x=y=z\ (QED)$$

1371. Mr. Bin entered the problem as follows:"

Let $a, b, c \ge 0$ such that $a+b+c=3$. Prove that:

$$a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}+15\ge\left(6-3\sqrt{3}\right)(ab+bc+ca)+9\sqrt{3}''$$

in the AI chat software, but the software could not solve it.

And you ? Let's solve this puzzle !

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables $= 0$ and WLOG we may assume $b = c = 0$

$(a = 3)$ and then : $LHS - RHS = 3^{\frac{2023}{2024}} + 15 - 9\sqrt{3} > \sqrt{3} + 15 - 9\sqrt{3} > 0$

$\boxed{\text{Case 2}}$ Exactly 1 variable $= 0$ and WLOG we may assume $a = 0$ $(b + c = 3)$

and then : $LHS - RHS = \sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\text{ terms}}.1} + \sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\text{ terms}}.1} + 15$

$$-9\left(\frac{6-3\sqrt{3}}{2}\right)\left(\frac{2bc}{(b+c)^2}\right) - 9\sqrt{3} \overset{G-H}{\geq} \frac{2024}{\frac{2023}{b}+1} + \frac{2024}{\frac{2023}{c}+1} + 15$$

$$-\frac{9(6-3\sqrt{3})}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right) - 9\sqrt{3}$$

$$= \frac{2024b^2}{2023b+b^2} + \frac{2024c^2}{2023c+c^2} + 15 - 27\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$-9\sqrt{3}\left(1-\frac{3}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right)\right) \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because b+c=3}}{\geq} \frac{2024(b+c)^2}{\frac{2023}{3}(b+c)^2+b^2+c^2} + 15$$

$$-27\left(\frac{2bc}{b^2+c^2+2bc}\right) - 9\sqrt{3}\left(1-\frac{3}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right)\right)$$

$$= \frac{2024(x+y)}{\frac{2023}{3}(x+y)+x} + 15 - 27\left(\frac{y}{x+y}\right) - 9\sqrt{3}\left(1-\frac{3}{2}\left(\frac{y}{x+y}\right)\right)$$

$$(x = b^2+c^2; y = 2bc) \overset{?}{\geq} 0 \Leftrightarrow \frac{3(6077x+6068y)(2x-y)}{(x+y)(2026x+2023y)} \underset{(*)}{\overset{?}{\geq}} \frac{9\sqrt{3}(2x-y)}{2(x+y)}$$

If $2x = y$, then : $b^2 + c^2 = bc \Rightarrow (b+c)^2 = 3bc \Rightarrow bc = 3$ and $\therefore b + c = 3$

$\therefore b + \frac{3}{b} = 3 \Rightarrow b^2 - 3b + 3 = 0$ and $\because \Delta = 9 - 12 < 0 \Rightarrow$ no real values of

b, c exist such that $b + c = 3$ and $b^2 + c^2 = bc \therefore 2x - y \neq 0$

$\therefore (*) \Leftrightarrow \frac{6077x+6068y}{2026x+2023y} \geq \frac{3\sqrt{3}}{2}$ and $\because \frac{3\sqrt{3}}{2} < \frac{13}{5} \therefore$ it suffices to prove :

$\frac{6077x+6068y}{2026x+2023y} > \frac{13}{5} \Leftrightarrow 3(1349x + 1347y) > 0 \to$ true $\because x, y > 0$

$\Rightarrow (*)$ is true $\therefore a^{\frac{2023}{2024}} + b^{\frac{2023}{2024}} + c^{\frac{2023}{2024}} + 15 > (6 - 3\sqrt{3})(ab + bc + ca) + 9\sqrt{3}$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and then : $LHS - RHS \overset{\because a+b+c=3}{=} \sqrt[2024]{\underbrace{a.a.a\ldots a}_{2023\text{ terms}}.1}$

$$+\sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\text{ terms}}.1} + \sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\text{ terms}}.1} + 15 - \frac{9(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2} - 9\sqrt{3}$$

$$\overset{G-H}{\geq} \frac{2024}{\frac{2023}{a}+1}+\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}+15-\frac{54(ab+bc+ca)}{(a+b+c)^2}$$
$$-9\sqrt{3}\left(1-\frac{3(ab+bc+ca)}{(a+b+c)^2}\right)$$
$$=\frac{2024a^2}{2023a+a^2}+\frac{2024b^2}{2023b+b^2}+\frac{2024c^2}{2023c+c^2}+15-\frac{54(ab+bc+ca)}{(a+b+c)^2}$$
$$-9\sqrt{3}\left(1-\frac{3(ab+bc+ca)}{(a+b+c)^2}\right)\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because a+b+c=3}}{\geq}\frac{2024(a+b+c)^2}{\frac{2023}{3}(a+b+c)^2+a^2+b^2+c^2}$$
$$+15-\frac{54(ab+bc+ca)}{(a+b+c)^2}-9\sqrt{3}\left(1-\frac{3(ab+bc+ca)}{(a+b+c)^2}\right)$$
$$=\frac{2024(u+2v)}{\frac{2023}{3}(u+2v)+u}+15-\frac{54v}{u+2v}-9\sqrt{3}\left(1-\frac{3v}{u+2v}\right)\left(u=\sum_{cyc}a^2, v=\sum_{cyc}ab\right)$$
$$\overset{?}{\geq} 0=\frac{3(6077u+12136v)(u-v)}{(u+2v)(1013u+2023v)}\overset{?}{\geq}\frac{9\sqrt{3}(u-v)}{u+2v}$$
$$\Leftrightarrow 6077u+12136v\overset{?}{\geq}3\sqrt{3}(1013u+2023v)\ (\because u-v\geq 0)\text{ and }\because 3\sqrt{3}<\frac{16}{5}$$
$$\therefore \text{it suffices to prove}: 6077u+12136v>\frac{16}{5}(1013u+2023v)$$
$$\Leftrightarrow 14177u+28312v>0\rightarrow \text{true}\because u,v>0$$
$$\therefore a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}+15\geq\left(6-3\sqrt{3}\right)(ab+bc+ca)+9\sqrt{3}$$
$$\therefore \text{combining all cases}, a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}+15$$
$$\geq\left(6-3\sqrt{3}\right)(ab+bc+ca)+9\sqrt{3}\ \forall a,b,c\geq 0\,\middle|\, a+b+c=3,$$
$$''='' \text{ iff } a=b=c=1\ (QED)$$

1372. Let $a,b,c\geq 0$ such that $a+b+c=3$. Prove that :

$$\text{(i) } a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}+2\left(16-9\sqrt{3}\right)\geq\left(6-3\sqrt{3}\right)(ab+bc+ca)\text{ and}$$

$$\text{(ii) } a^{\frac{2024}{2023}}+b^{\frac{2024}{2023}}+c^{\frac{2024}{2023}}+2\left(16-9\sqrt{3}\right)\geq\left(6-3\sqrt{3}\right)(ab+bc+ca)$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove :

$$\text{(i) } a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}+2\left(16-9\sqrt{3}\right)\geq\left(6-3\sqrt{3}\right)(ab+bc+ca)$$

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$ $(a=3)$

and then : $LHS-RHS=3^{\frac{2023}{2024}}+2\left(16-9\sqrt{3}\right)>0\therefore$ (i) is true (strict inequality)

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $(b+c=3)$

$$and\ then: \text{LHS} - \text{RHS} = \sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\ terms}.1} + \sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\ terms}.1} + 2(16-9\sqrt{3})$$

$$-9\left(\frac{6-3\sqrt{3}}{2}\right)\left(\frac{2bc}{(b+c)^2}\right) \overset{G-H}{\geq} \frac{2024}{\frac{2023}{b}+1} + \frac{2024}{\frac{2023}{c}+1} + 2(16-9\sqrt{3})$$

$$-\frac{9(6-3\sqrt{3})}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$= \frac{2024b^2}{2023b+b^2} + \frac{2024c^2}{2023c+c^2} + 32 - 27\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$-18\sqrt{3} + \frac{27\sqrt{3}}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right) \overset{\substack{Bergstrom\\ and\\ \because b+c=3}}{\geq}$$

$$\frac{2024(b+c)^2}{\frac{2023}{3}(b+c)^2 + b^2 + c^2} + 32 - 27\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$-18\sqrt{3} + \frac{27\sqrt{3}}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$= \frac{2024(x+y)}{\frac{2023}{3}(x+y)+x} + 32 - 27\left(\frac{y}{x+y}\right) - 18\sqrt{3} + \frac{27\sqrt{3}}{2}\left(\frac{y}{x+y}\right)$$

$$(x = b^2+c^2; y = 2bc) = \frac{70904x^2 + 87010xy + 16187y^2}{(x+y)(2026x+2023y)} - \frac{9\sqrt{3}(4x+y)}{2(x+y)}$$

$$\overset{\frac{9\sqrt{3}}{2}<8}{>} \frac{70904x^2 + 87010xy + 16187y^2}{(x+y)(2026x+2023y)} - \frac{8(4x+y)}{x+y} = \frac{6072x^2 + 6066xy + 3y^2}{(x+y)(2026x+2023y)}$$

$> 0 \because x \geq y > 0 \therefore$ (i) is true (strict inequality)

$\boxed{Case\ 3}$ $a, b, c > 0$ and then : LHS − RHS $\overset{\because a+b+c=3}{=}$

$$\sqrt[2024]{\underbrace{a.a.a\ldots a}_{2023\ terms}.1} + \sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\ terms}.1} + \sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\ terms}.1}$$

$$+2(16-9\sqrt{3}) - \frac{9(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{G-H}{\geq} \frac{2024}{\frac{2023}{a}+1} + \frac{2024}{\frac{2023}{b}+1} + \frac{2024}{\frac{2023}{c}+1} + 2(16-9\sqrt{3}) - \frac{9(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}$$

$$= \frac{2024a^2}{2023a+a^2} + \frac{2024b^2}{2023b+b^2} + \frac{2024c^2}{2023c+c^2} + 2(16-9\sqrt{3})$$

$$-\frac{9(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2} \overset{\substack{Bergstrom\\ and\\ \because a+b+c=3}}{\geq}$$

$$\frac{2024(a+b+c)^2}{\frac{2023}{3}(a+b+c)^2 + a^2+b^2+c^2} + 2(16-9\sqrt{3}) - \frac{9(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}$$

$$= \frac{2024(u+2v)}{\frac{2023}{3}(u+2v)+u} + 32 - \frac{54v}{u+2v} - 18\sqrt{3} + 27\sqrt{3}\left(\frac{v}{u+2v}\right)$$

$$\left(u=\sum_{cyc}a^2, v=\sum_{cyc}ab\right)=\frac{70904u^2+174020uv+64748v^2}{(u+2v)(2026u+4046v)}-\frac{9\sqrt{3}(2u+v)}{u+2v}$$

$$\overset{9\sqrt{3}<16}{>}\frac{70904u^2+174020uv+64748v^2}{(u+2v)(2026u+4046v)}-\frac{16(2u+v)}{u+2v}$$

$$=\frac{6072u^2+12132uv+12v^2}{(u+2v)(2026u+4046v)}>0 \because u\geq v>0 \therefore \text{(i) is true (strict inequality)}$$

$\therefore$ combining *all* cases, (i) is true (strict inequality) $\forall\, a,b,c\geq 0 \mid a+b+c=3$

We shall now prove :

$$\text{(ii) } a^{\frac{2024}{2023}}+b^{\frac{2024}{2023}}+c^{\frac{2024}{2023}}+2(16-9\sqrt{3})\geq(6-3\sqrt{3})(ab+bc+ca)$$

Case 1 Exactly 2 variables = 0 and WLOG we may assume $b=c=0$ $(a=3)$

and then : $LHS-RHS=3^{\frac{2024}{2023}}+2(16-9\sqrt{3})>0 \therefore$ (ii) is true (strict inequality)

Case 2 Exactly 1 variable = 0 and WLOG we may assume $a=0$ $(b+c=3)$

and then : $LHS-RHS=b^{\frac{2024}{2023}}+c^{\frac{2024}{2023}}+2(16-9\sqrt{3})-(6-3\sqrt{3})(bc)$

$$\overset{\text{Power-Mean inequality and A-G}}{\geq}\frac{1}{2^{\frac{2024}{2023}-1}}(b+c)^{\frac{2024}{2023}}+2(16-9\sqrt{3})-\frac{(6-3\sqrt{3})}{4}(b+c)^2$$

$$\overset{b+c=3}{=}\frac{1}{2^{\frac{1}{2023}}}.3^{\frac{2024}{2023}}+2(16-9\sqrt{3})-\frac{9(6-3\sqrt{3})}{4}=\frac{1}{2^{\frac{1}{2023}}}.3^{\frac{2024}{2023}}+\frac{37}{2}-\frac{45\sqrt{3}}{4}$$

$$>\frac{3}{2}+\frac{37}{2}-\frac{45\sqrt{3}}{4}>\frac{1}{2}>0 \therefore \text{(ii) is true (strict inequality)}$$

Case 3 $a,b,c>0$ and then : $LHS-RHS\overset{\text{Power-Mean inequality and A-G}}{\geq}\frac{1}{3^{\frac{2024}{2023}-1}}\left(\sum_{cyc}a\right)^{\frac{2024}{2023}}$

$$+2(16-9\sqrt{3})-(2-\sqrt{3})(a+b+c)^2\overset{a+b+c=3}{=}\frac{3^{\frac{2024}{2023}}}{3^{\frac{1}{2023}}}+2(16-9\sqrt{3})$$

$$-9(2-\sqrt{3})=3+32-18\sqrt{3}-18+9\sqrt{3}>17-9\sqrt{3}>0\ (\because 289>243)$$

$\therefore$ (ii) is true (strict inequality) $\therefore$ combining *all* cases,

(ii) is true (strict inequality) $\forall\, a,b,c\geq 0 \mid a+b+c=3$ (QED)

1373. Let $a,b,c\geq 0$ such that : $a+b+c=3$. Prove that :

a) $9(a^k+b^k+c^k)+ab+bc+ca\leq 27+3abc$ with $k=\frac{1}{2}$ and

b) $9(a^k+b^k+c^k)+ab+bc+ca\geq 27+3abc$ with $k\geq 1$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

When exactly 2 variables = 0 (WLOG $b=c=0$ and $a=3$), then :

$$\text{LHS of } a) = 9\sqrt{3} < 27 = \text{RHS of } a) \text{ and LHS of b}) = 9.3^{k} \overset{k\geq 1}{\geq} 27 = \text{RHS of b})$$

When exactly 1 variable $= 0$ (WLOG and $a = 0$ and $b + c = 3; b, c > 0$), then :

$$\text{LHS of } a) = 9\left(\sqrt{b}+\sqrt{c}\right) + bc \overset{\substack{\text{CBS}\\ \text{and}\\ \text{A-G}}}{\leq} 9\sqrt{2(b+c)} + \frac{(b+c)^2}{4} \overset{b+c=3}{=} 9\sqrt{6} + \frac{9}{4} < 27$$

$$= \text{RHS of } a) \text{ and LHS of b}) = 9\left(\left(1+(b-1)\right)^{k} + \left(1+(c-1)\right)^{k}\right) + bc$$

$$\overset{\text{via Bernoulli} \because k\geq 1}{\geq} 9\left(2 + k\left((b-1)+(c-1)\right)\right) + bc \overset{b+c=3}{=} 18 + 9k(3-2) + bc$$

$$\overset{k\geq 1}{\geq} 27 + bc > 27 = RHS\ of\ b)$$

We now consider the case when $a, b, c > 0$ and $a) \Leftrightarrow$

$$9\sum_{cyc} x + \sum_{cyc} x^2y^2 \leq 27 + 3x^2y^2z^2 \quad \left(x = \sqrt{a}, y = \sqrt{b}, z = \sqrt{c}\right)$$

$$\overset{3=\sum_{cyc} x^2}{\Leftrightarrow} \left(\sum_{cyc} x^2\right)^3 + 3x^2y^2z^2 - \frac{1}{3}\left(\sum_{cyc} x^2y^2\right)\left(\sum_{cyc} x^2\right) \geq 9\sum_{cyc} x$$

$$\overset{\because 3=\sum_{cyc} x^2}{\Leftrightarrow} 3\left(\sum_{cyc} x^2\right)^3 + 9x^2y^2z^2 - \left(\sum_{cyc} x^2y^2\right)\left(\sum_{cyc} x^2\right) \geq \sqrt{3\left(\sum_{cyc} x^2\right)^5}\sum_{cyc} x$$

$$\Leftrightarrow \boxed{\left(3\left(\sum_{cyc} x^2\right)^3 + 9x^2y^2z^2 - \left(\sum_{cyc} x^2y^2\right)\left(\sum_{cyc} x^2\right)\right)^2 \overset{(*)}{\geq} 3\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^2\right)^5}$$

$$\left(\begin{array}{c} \because 3\left(\sum_{cyc} a\right)^3 + 9abc - \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right) \\ 3\sum_{cyc} a^3 + 9\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - abc\right) + 9abc - \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \\ 3\sum_{cyc} a^3 + 9\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - abc\right) + 9abc - \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \end{array}\right)$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y$
$\Rightarrow$ X, Y, Z form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \rightarrow (1)$

$\Rightarrow x = s - X, y = s - Y, z = s - Z$ *and* such substitutions $\Rightarrow$

$$\sum_{cyc} xy = \sum_{cyc} (s-X)(s-Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (2) \text{ and}$$

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via (1) and (2)}}{=}$$

$$(4Rr + r^2)^2 - 2\left(\prod_{cyc}(s-X)\right).s = (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} x^2y^2 = r^2\left((4R+r)^2 - 2s^2\right) \to (3) \text{ and}$$

$$\sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (1) and (2)}}{=} s^2 - (4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \to (4)$$

$\therefore$ via (1)(3) *and* (4), $(*)$

$$\Leftrightarrow \boxed{\begin{gathered}\left(3(s^2 - 8Rr - 2r^2)^3 + 9r^4s^2 - r^2\left((4R+r)^2 - 2s^2\right)(s^2 - 8Rr - 2r^2)\right)^2 \\ -3s^2(s^2 - 8Rr - 2r^2)^5 \overset{(**)}{\geq} 0\end{gathered}}$$

Now, $43,008t^3 - 126,400t^2 + 124,192t - 40,192$

$$\left(\text{where } t = \frac{R}{r} \overset{\text{Euler}}{\geq} 2 \textit{ and comma represents the thousands} - \textit{separator}\right)$$

$$= (t-2)(43,008t^2 - 40,384t + 43,424) + 46,656 > 0$$

$$\Rightarrow 43,008R^3 - 126,400R^2r + 124,192Rr^2 - 40,192r^3 > 0 \text{ and}$$

$$209,152t^4 - 855,808t^3 + 1,311,424t^2 - 877,440t + 214,794$$

$$= (t-2)\left((t-2)(209,152t^2 - 19,200t + 398,016) + 791,424\right) + 205,578 > 0$$

$$\Rightarrow 209,152R^4 - 855,808R^3r + 1,311,424R^2r^2 - 877,440Rr^3 + 214,794r^4 > 0$$

$$\therefore \text{via Gerretsen}, 6(s^2 - 16Rr + 5r^2)^6 + (264Rr - 246r^2)(s^2 - 16Rr + 5r^2)^5$$

$$+r^2(4704R^2 - 8952Rr + 4276r^2)(s^2 - 16Rr + 5r^2)^4$$

$$+r^3(43,008R^3 - 126,400R^2r + 124,192Rr^2 - 40,192r^3)(s^2 - 16Rr + 5r^2)^3$$

$$+r^4\left(\begin{gathered}209,152R^4 - 855,808R^3r + 1,311,424R^2r^2 \\ -877,440Rr^3 + 214,794r^4\end{gathered}\right)(s^2 - 16Rr + 5r^2)^2 \overset{(\bullet)}{\geq} 0$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \geq$ LHS of $(\bullet)$

$$\Leftrightarrow \left(\begin{gathered}247,808R^5 - 1,361,664R^4r + 2,948,224R^3r^2 \\ -3,097,408R^2r^3 + 1,573,116Rr^4 - 308,477r^5\end{gathered}\right)s^2$$

$$\overset{(***)}{\geq} r\left(\begin{gathered}3,760,128R^6 - 21,463,040R^5r + 49,317,504R^4r^2 \\ -57,311,104R^3r^3 + 35,056,912R^2r^4 - 10,519,128Rr^5 + 1,171,433r^6\end{gathered}\right)$$

Now, $247,808t^5 - 1,361,664t^4 + 2,948,224t^3 - 3,097,408t^2 + 1,573,116t$
$-308,477$

$$= (t-2)\left((t-2)\begin{pmatrix}247{,}808t^3 - 370{,}432t^2\\ +475{,}264t + 285{,}376\end{pmatrix} + 813{,}564\right) + 177{,}147 > 0$$

$$\Rightarrow 247{,}808R^5 - 1{,}361{,}664R^4r + 2{,}948{,}224R^3r^2 - 3{,}097{,}408R^2r^3 + 1{,}573{,}116Rr^4$$

$$-308{,}477r^5 > 0 \therefore \text{LHS of } (***) \overset{\text{Rouche}}{\geq}$$

$$\begin{pmatrix}247{,}808R^5 - 1{,}361{,}664R^4r + 2{,}948{,}224R^3r^2\\ -3{,}097{,}408R^2r^3 + 1{,}573{,}116Rr^4 - 308{,}477r^5\end{pmatrix}\begin{pmatrix}2R^2 + 10Rr - r^2\\ -2(R-2r)\sqrt{R^2-2Rr}\end{pmatrix}$$

$$\overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 2(R-2r)\begin{pmatrix}247{,}808R^6 - 1{,}507{,}072R^5r + 3{,}733{,}376R^4r^2\\ -4{,}867{,}456R^3r^3 + 3{,}532{,}604R^2r^4 - 1{,}357{,}441Rr^5 + 215{,}739r^6\end{pmatrix}$$

$$\underset{(****)}{\overset{?}{\geq}} 2(R-2r)\sqrt{R^2-2Rr}\begin{pmatrix}247{,}808R^5 - 1{,}361{,}664R^4r + 2{,}948{,}224R^3r^2\\ -3{,}097{,}408R^2r^3 + 1{,}573{,}116Rr^4 - 308{,}477r^5\end{pmatrix}$$

Again, $247{,}808t^6 - 1{,}507{,}072t^5 + 3{,}733{,}376t^4 - 4{,}867{,}456t^3 + 3{,}532{,}604t^2$
$-1{,}357{,}441t + 215{,}739$

$$= (t-2)\left((t-2)\left((t-2)\begin{pmatrix}247{,}808t^3 - 20{,}224t^2\\ +638{,}336t + 1{,}187{,}712\end{pmatrix} + 2{,}837{,}052\right) + 844{,}911\right)$$

$$+59{,}049 > 0 \Rightarrow 247{,}808R^6 - 1{,}507{,}072R^5r + 3{,}733{,}376R^4r^2 - 4{,}867{,}456R^3r^3$$

$$+3{,}532{,}604R^2r^4 - 1{,}357{,}441Rr^5 + 215{,}739r^6 > 0 \text{ and } \because R - 2r \overset{\text{Euler}}{\geq} 0$$

$\therefore$ in order to prove $(****)$, it suffices to prove :

$$\begin{pmatrix}247{,}808R^6 - 1{,}507{,}072R^5r + 3{,}733{,}376R^4r^2 -\\ 4{,}867{,}456R^3r^3 + 3{,}532{,}604R^2r^4 - 1{,}357{,}441Rr^5 + 215{,}739r^6\end{pmatrix}^2$$

$$> (R^2 - 2Rr)\begin{pmatrix}247{,}808R^5 - 1{,}361{,}664R^4r +\\ 2{,}948{,}224R^3r^2 - 3{,}097{,}408R^2r^3 + 1{,}573{,}116Rr^4 - 308{,}477r^5\end{pmatrix}^2$$

$$\Leftrightarrow 50{,}751{,}078{,}400t^{11} - 543{,}453{,}872{,}128t^{10} + 2{,}529{,}415{,}659{,}520t^9$$
$$-6{,}675{,}031{,}293{,}952t^8 + 10{,}849{,}957{,}036{,}032t^7 - 10{,}843{,}569{,}209{,}344t^6$$
$$+5{,}808{,}184{,}095{,}232t^5 - 209{,}088{,}406{,}656t^4 - 1{,}948{,}958{,}574{,}656t^3$$
$$+1{,}330{,}648{,}500{,}336t^2 - 395{,}389{,}808{,}740t + 46{,}543{,}316{,}121 > 0$$

$$\Leftrightarrow (t-2)\left((t-2)\left((t-2)\left((t-2)\begin{pmatrix}(t-2)(\lambda) +\\ 7{,}520{,}970{,}248{,}064\end{pmatrix} + 2{,}467{,}147{,}211{,}712\right)\right.\right.$$

$$\left.\left. + 441{,}062{,}017{,}776\right) + 37{,}020{,}180{,}060\right) + 3{,}486{,}784{,}401 > 0$$

$$\begin{pmatrix}\text{where } \lambda = 50{,}751{,}078{,}400t^6 - 35{,}943{,}088{,}128t^5 + 139{,}941{,}642{,}240t^4 +\\ 222{,}194{,}925{,}568t^3 + 538{,}707{,}279{,}872t^2 + 1{,}350{,}519{,}504{,}896t + 3{,}195{,}171{,}795{,}456\end{pmatrix}$$

$\rightarrow$ true $\therefore t \geq 2$ and $\therefore \lambda > 0 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore 9(a^k + b^k + c^k) + ab + bc + ca \leq 27 + 3abc \text{ for } k = \frac{1}{2}\ \forall a,b,c > 0 \,\Big|\, \sum_{\text{cyc}} a = 3$$

Also, $9(a^k + b^k + c^k) + ab + bc + ca - 27 - 3abc$

$$= 9\sum_{\text{cyc}}(1 + (a-1))^k + \sum_{\text{cyc}} ab - 27 - 3abc \overset{\text{via Bernoulli} \because k \geq 1}{\geq}$$

$$9\left(3+k\left(\sum_{cyc}a-3\right)\right)+\sum_{cyc}ab-27-3abc\overset{a+b+c=3}{=}$$

$$\sum_{cyc}ab-3abc\overset{A-G}{\geq}3\sqrt[3]{a^2b^2c^2}-3abc\overset{?}{\geq}0\Leftrightarrow a^2b^2c^2\overset{?}{\geq}a^3b^3c^3\Leftrightarrow abc\overset{?}{\leq}1$$

$$\rightarrow \text{true} \because 3=\sum_{cyc}a\overset{A-G}{\geq}3\sqrt[3]{abc}\Rightarrow abc\leq 1$$

$$\therefore 9(a^k+b^k+c^k)+ab+bc+ca\geq 27+3abc \text{ with } k\geq 1\ \forall a,b,c>0 \Big| \sum_{cyc}a=3$$

$\therefore$ combining *all* cases,

$$a)\ 9(a^k+b^k+c^k)+ab+bc+ca\leq 27+3abc \text{ with } k=\frac{1}{2}$$

$$\forall a,b,c\geq 0\,\big|\, a+b+c=3,''=''\text{ iff } a=b=c=1$$

$$b)\ 9(a^k+b^k+c^k)+ab+bc+ca\geq 27+3abc \text{ with } k\geq 1$$

$$\forall a,b,c\geq 0\,\big|\, a+b+c=3,''=''\text{ iff } (a=b=c=1) \text{ or}, (a=3,b=c=0;k=1)$$

$$\text{or}, (b=3,c=a=0;k=1) \text{ or}, (c=3,a=b=0;k=1)\ (\text{QED})$$

1374. Let $a,b,c\geq 0$ such that $a+b+c=4$. Prove that :

(i) $a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}+2(16-9\sqrt{3})\geq(6-3\sqrt{3})(ab+bc+ca)$ and

(ii) $a^{2023}+b^{2023}+c^{2023}+2(16-9\sqrt{3})\geq(6-3\sqrt{3})(ab+bc+ca)$

Proposed by Nguyen Van Canh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

We shall first prove :

$$\text{(i) } a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}+2(16-9\sqrt{3})\geq(6-3\sqrt{3})(ab+bc+ca)$$

$\boxed{\text{Case 1}}$ *Exactly* 2 variables $=0$ *and* WLOG *we may assume* $b=c=0$ $(a=4)$

and then : $\text{LHS}-\text{RHS}=4^{\frac{2024}{2025}}+2(16-9\sqrt{3})>0$ $\therefore$ (i) is true (strict inequ*ality*)

$\boxed{\text{Case 2}}$ *Exactly* 1 variable $=0$ *and* WLOG *we may assume* $a=0$ $(b+c=4)$ *and*

$$\text{then : LHS}-\text{RHS}=\sqrt[2025]{\underbrace{b.b.b\ldots b}_{2024\text{ terms}}.1}+\sqrt[2025]{\underbrace{c.c.c\ldots c}_{2024\text{ terms}}.1}+2(16-9\sqrt{3})$$

$$-16\left(\frac{6-3\sqrt{3}}{2}\right)\left(\frac{2bc}{(b+c)^2}\right)\overset{G-H}{\geq}$$

$$\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}+2(16-9\sqrt{3})-8(6-3\sqrt{3})\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$=\frac{2025b^2}{2024b+b^2}+\frac{2025c^2}{2024c+c^2}+32-48\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$-18\sqrt{3}+24\sqrt{3}\left(\frac{2bc}{b^2+c^2+2bc}\right)\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because b+c=4}}{\geq}$$

$$\frac{2025(b+c)^2}{506(b+c)^2+b^2+c^2}+32-48\left(\frac{2bc}{b^2+c^2+2bc}\right)$$
$$-18\sqrt{3}+24\sqrt{3}\left(\frac{2bc}{b^2+c^2+2bc}\right)$$
$$=\frac{2025(x+y)}{506(x+y)+x}+32-48\left(\frac{y}{x+y}\right)-18\sqrt{3}+24\sqrt{3}\left(\frac{y}{x+y}\right)$$
$$\left(x=b^2+c^2;y=2bc\right)=\frac{18249x^2+12130xy-6071y^2}{(x+y)(507x+506y)}-\frac{6\sqrt{3}(3x-y)}{x+y}$$
$$\overset{6\sqrt{3}<11}{>}\frac{18249x^2+12130xy-6071y^2}{(x+y)(507x+506y)}-\frac{11(3x-y)}{x+y}$$
$$=\frac{1518x^2+1009xy-505y^2}{(x+y)(507x+506y)}=\frac{(506x+505y)(3x-y)}{(x+y)(507x+506y)}>0\because x\geq y>0$$
$\therefore$ (i) is true (strict inequality)

Case 3 $a,b,c>0$ and then : LHS − RHS $\overset{\because a+b+c=4}{=}$

$$\sqrt[2025]{\underbrace{a.a.a\ldots a}_{2024\text{ terms}}.1}+\sqrt[2025]{\underbrace{b.b.b\ldots b}_{2024\text{ terms}}.1}+\sqrt[2025]{\underbrace{c.c.c\ldots c}_{2024\text{ terms}}.1}+2\left(16-9\sqrt{3}\right)$$
$$-\frac{16\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}$$
$$\overset{G-H}{\geq}\frac{2025}{\frac{2024}{a}+1}+\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}+2\left(16-9\sqrt{3}\right)-\frac{16\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}$$
$$=\frac{2025a^2}{2024a+a^2}+\frac{2025b^2}{2024b+b^2}+\frac{2025c^2}{2024c+c^2}+2\left(16-9\sqrt{3}\right)$$
$$-\frac{16\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}\overset{\text{Bergstrom and }\because a+b+c=4}{\geq}$$
$$\frac{2025(a+b+c)^2}{506(a+b+c)^2+a^2+b^2+c^2}+2\left(16-9\sqrt{3}\right)-\frac{16\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}$$
$$=\frac{2025(u+2v)}{506u+1012v+u}+32-\frac{96v}{u+2v}-18\sqrt{3}+48\sqrt{3}\left(\frac{v}{u+2v}\right)$$
$$\left(u=\sum_{cyc}a^2,v=\sum_{cyc}ab\right)=\frac{18249u^2+24260uv-24284v^2}{(u+2v)(507u+1012v)}-\frac{6\sqrt{3}(3u-2v)}{u+2v}$$
$$\overset{6\sqrt{3}<11}{>}\frac{18249u^2+24260uv-24284v^2}{(u+2v)(507u+1012v)}-\frac{11(3u-2v)}{u+2v}$$
$$=\frac{1518u^2+2018uv-2020v^2}{(u+2v)(507u+1012v)}=\frac{2(3u-2v)(253u+505v)}{(u+2v)(507u+1012v)}>0\because u\geq v>0$$
$\therefore$ (i) is true (strict inequality) $\therefore$ combining all cases,

(i) is true (strict inequality) $\forall\, a,b,c\geq 0\mid a+b+c=4$

We shall now prove :

(ii) $a^{2023}+b^{2023}+c^{2023}+2\left(16-9\sqrt{3}\right)\geq\left(6-3\sqrt{3}\right)(ab+bc+ca)$

Case 1 Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$ $(a=4)$

$$\text{and then : LHS} - \text{RHS} = 4^{2023} + 2(16 - 9\sqrt{3}) > 0 \therefore \text{(ii) is true (strict inequality)}$$

$$\boxed{\text{Case 2}}\ \text{Exactly 1 variable} = 0 \text{ and WLOG we may assume } a = 0\ (b + c = 4)$$

$$\text{and then : LHS} - \text{RHS} = b^{2023} + c^{2023} + 2(16 - 9\sqrt{3}) - (6 - 3\sqrt{3})(bc)$$

$$\overset{\substack{\text{Holder}\\ \text{and}\\ \text{A-G}}}{\geq} \frac{1}{2^{2022}}(b + c)^{2023} + 2(16 - 9\sqrt{3}) - \frac{(6 - 3\sqrt{3})}{4}(b + c)^2 \overset{b+c=4}{=}$$

$$2^{2024} + 2(16 - 9\sqrt{3}) - 4(6 - 3\sqrt{3}) = 2^{2024} + 8 - 6\sqrt{3} > 0$$

$$\therefore \text{(ii) is true (strict inequality)}$$

$$\boxed{\text{Case 3}}\ a, b, c > 0 \text{ and then : LHS} - \text{RHS} \overset{\substack{\text{Holder}\\ \text{and}\\ \text{A-G}}}{\geq} \frac{1}{3^{2022}}(a + b + c)^{2023}$$

$$+2(16 - 9\sqrt{3}) - (2 - \sqrt{3})(a + b + c)^2 \overset{a+b+c=4}{=} \frac{4^{2023}}{3^{2022}} + 2(16 - 9\sqrt{3})$$

$$-16(2 - \sqrt{3}) > \left(\frac{4}{3}\right)^{2022} - 2\sqrt{3} > \left(\frac{4}{3}\right)^{5} - 4 = 4\left(\frac{256}{243} - 1\right) > 0$$

$$\therefore \text{(ii) is true (strict inequality)} \therefore \text{combining all cases,}$$

$$\text{(ii) is true (strict inequality) } \forall\, a, b, c \geq 0 \mid a + b + c = 4 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $ab + bc + ca \leq \frac{(a + b + c)^2}{3} = \frac{16}{3}$, so it suffices to prove that

$$a^{\frac{2024}{2025}} + b^{\frac{2024}{2025}} + c^{\frac{2024}{2025}} \geq 2\sqrt{3} \quad \text{and} \quad a^{2023} + b^{2023} + c^{2023} \geq 2\sqrt{3}.$$

i) Since $\frac{a}{4}, \frac{b}{4}, \frac{c}{4} \leq 1$, then $\left(\frac{a}{4}\right)^{\frac{2024}{2025}} \geq \frac{a}{4}$, $\left(\frac{b}{4}\right)^{\frac{2024}{2025}} \geq \frac{b}{4}$, $\left(\frac{c}{4}\right)^{\frac{2024}{2025}} \geq \frac{c}{4}$, thus

$$a^{\frac{2024}{2025}} + b^{\frac{2024}{2025}} + c^{\frac{2024}{2025}} \geq 4^{\frac{2024}{2025}} . \frac{a + b + c}{4} = 4^{\frac{2024}{2025}} = 2.2^{1-\frac{2}{2025}} > 2.2^{1-\frac{1}{5}} = 2\sqrt[5]{16}$$

$$> 2\sqrt{3},$$

which completes the proof of *i*).

ii) By Power Mean inequality, we have

$$a^{2023} + b^{2023} + c^{2023} \geq 3\left(\frac{a + b + c}{3}\right)^{2023} = \frac{4^{2023}}{3^{2022}} = \frac{2^{4046}}{\sqrt{3}^{4044}} > 2.\frac{\sqrt{3}^{4045}}{\sqrt{3}^{4044}} = 2\sqrt{3},$$

which completes the proof of *ii*).

1375. Let $a, b, c \geq 0$ such that $a + b + c = 3$. Prove that

$$a)\ 9(\sqrt{a} + \sqrt{b} + \sqrt{c}) + ab + bc + ca \leq 27 + 3abc.$$

$$b)\ 9(a^k + b^k + c^k) + ab + bc + ca \geq 27 + 3abc, \quad \text{with } k \geq 1.$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

a) Let $x := \sqrt{a},\ \ y := \sqrt{b},\ \ z := \sqrt{c}$ and $p := x+y+z,\ \ q := xy+yz+zx,\ \ r := xyz$.
The given condition is equivalent to $x^2+y^2+z^2 = p^2-2q$
$= 3$, and the problem becomes,

$$9(x+y+z)+x^2y^2+y^2z^2+z^2x^2 \le 27+3x^2y^2z^2 \ \Leftrightarrow\ 9p+q^2-2pr \le 27+3r^2$$

$$\overset{2q=p^2-3}{\Longleftrightarrow}\quad 36p+(p^2-3)^2-8pr \le 108+12r^2$$

$$\Leftrightarrow\ 99-36p+6p^2-p^4+8pr+12r^2 \ge 0\ \ (*)$$

We have $3 = x^2+y^2+z^2 \le p^2 \le 3(x^2+y^2+z^2) = 9$, then $\sqrt{3} \le p \le 3$.

If $\sqrt{3} \le p \le \sqrt{6}$, we have $LHS_{(*)} \overset{r\ge 0}{\ge} 9(11-4p)+p^2(6-p^2) \ge 0$.

If $\sqrt{3} \le p \le 3$, we have

$$0 \le (a-b)^2(b-c)^2(c-a)^2 = \frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}$$

$$\Rightarrow\ r \ge \frac{-2p^3+9pq-2\sqrt{(p^2-3q)^3}}{27} \overset{2q=p^2-3}{\cong} \frac{5p^3-27p-\sqrt{2(9-p^2)^3}}{54}.$$

$$\Rightarrow\ LHS_{(*)} \ge 105-36p+3p^2-\frac{31p^4}{27}+\frac{23p^6}{243}-\frac{2p(5p^2-9)\sqrt{2(9-p^2)^3}}{243}$$

$$= (3-p)\left(35-\frac{p}{3}+\frac{8p^2}{9}+\frac{8p^3}{27}-\frac{23p^4}{81}-\frac{23p^5}{243}-\frac{2p(5p^2-9)\sqrt{2(3+p)^3(3-x)}}{243}\right)$$

$$\overset{AM-GM}{\ge} (3-p)\left(35-\frac{p}{3}+\frac{8p^2}{9}+\frac{8p^3}{27}-\frac{23p^4}{81}-\frac{23p^5}{243}\right.$$
$$\left.-\frac{p(5p^2-9)[2(3+p)^2(3-p)+(3+p)]}{243}\right)$$

$$= (3-p)\left[(3-p)\left[\frac{35}{3}+\frac{121p}{27}+(3-p)\left(\frac{164p^2}{243}+\frac{239p^3}{729}+\frac{44p^4}{729}\right)+\frac{46p^5}{2187}\right]\right.$$
$$\left.+\frac{4p^6}{2187}\right] \overset{3\ge p}{\ge} 0,$$

which completes the proof of a). Equality holds iff $a=b=c=1$.
b) Since $k \ge 1$ then by Power Mean inequality, we have

$$a^k+b^k+c^k \ge 3\left(\frac{a+b+c}{3}\right)^k = 3,$$

and by AM − GM inequality, we have

$$ab+bc+ca = \frac{(a+b+c)(ab+bc+ca)}{3} \ge 3abc.$$

Using these two inequalities, we have

$$9(a^k+b^k+c^k)+ab+bc+ca \ge 27+3abc.$$

So the proof is complete. Equality holds iff
$a=b=c=1$ or $k=1, a=3, b=c=0$ and permutations.

1376. If $a,b,c\geq 0$ such that : $ab+bc+ca>0$. Prove that :

$$\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2}\geq\frac{2(a^2+b^2+c^2)}{a+b+c}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly one variable $=0$ and WLOG we may assume $a=0$ and

$$\text{then}:\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2}=\frac{b^4}{b^2c}+\frac{c^4}{c^2b}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{(b^2+c^2)^2}{bc(b+c)}\overset{A-G}{\geq}\frac{2bc(b^2+c^2)}{bc(b+c)}=\frac{2(a^2+b^2+c^2)}{a+b+c}$$

$$\therefore\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2}\geq\frac{2(a^2+b^2+c^2)}{a+b+c}$$

$\boxed{\text{Case 2}}$ $a,b,c>0$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

$$\text{yielding }2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab$$

$$\overset{\text{via (1) and (3)}}{=}s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4),$$

$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{\text{via (1),(2) and (3)}}{=}(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}a^2b^2=r^2\left((4R+r)^2-2s^2\right)\to(5),$$

$$\sum_{cyc}a^3=\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)\overset{\text{via (1)}}{=}s^3-3.4Rrs$$

$$\Rightarrow\sum_{cyc}a^3=s(s^2-12Rr)\to(6)\text{ and }\sum_{cyc}a^4=\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2\overset{\text{via (4) and (5)}}{=}$$

$$(s^2-8Rr-2r^2)^2-2r^2\left((4R+r)^2-2s^2\right)$$

$$\Rightarrow \sum_{cyc} a^4 = \left(s^2 - 8Rr - 2r^2\right)^2 - 2r^2\left((4R+r)^2 - 2s^2\right) \to (7)$$

Now, $(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$

$$= \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + 2abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 6a^2b^2c^2$$

$$+\left(\sum_{cyc} ab\right)^3 - 3r^2s.4Rrs \overset{via\ (1),(2),(3),(4),(5)\ and\ (6)}{=}$$

$$r^2\left(s^2-8Rr-2r^2\right)\left((4R+r)^2-2s^2\right) + r^2s^2\left(s^2-12Rr\right) + 2r^2s^2\left(4Rr+r^2\right)$$

$$-6r^4s^2 + \left(4Rr+r^2\right)^3 - 12Rr^3s^2$$

$$\therefore \boxed{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \overset{(*)}{=} r^2\left(s^2(4R+r)^2 - s^4 - r(4R+r)^3\right)}$$

$$\text{Also, } \left(\sum_{cyc} a\right)\left(\sum_{cyc}\left(a^2(b+c)(c^2+ca+a^2)(a^2+ab+b^2)\right)\right)$$

$$= a^2(b+c)\left(\sum_{cyc} a^2b^2 + (a^3+abc)\left(\sum_{cyc} a\right)\right)$$

$$= \left(\sum_{cyc} a^2b^2 + abc\left(\sum_{cyc} a\right)\right)\left(\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2 - 3abc\left(\sum_{cyc} a\right)\right)$$

$$+\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^4\right)\left(\sum_{cyc} a\right)^2 - abc\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a\right)^2$$

$$\overset{via\ (1),(2),(3),(5),(6)\ and\ (7)}{=} \left(r^2\left((4R+r)^2-2s^2\right)+r^2s^2\right)\left(\left(4Rr+r^2\right)s^2 - 3r^2s^2\right)$$

$$+\left(4Rr+r^2\right)\left(\left(s^2-8Rr-2r^2\right)^2 - 2r^2\left((4R+r)^2-2s^2\right)\right)s^2 - r^2s^4\left(s^2-12Rr\right)$$

$$\Rightarrow \boxed{\left(\sum_{cyc} a\right)\left(\sum_{cyc}\left(a^2(b+c)(c^2+ca+a^2)(a^2+ab+b^2)\right)\right)}$$

$$\overset{(**)}{=} 2rs^2\left(2Rs^4 - rs^2\left(32R^2+4Rr-r^2\right) + 6Rr^2(4R+r)^2\right)$$

$$\therefore (*), (**) \Rightarrow \frac{a^2(b+c)}{b^2+bc+c^2} + \frac{b^2(c+a)}{c^2+ca+a^2} + \frac{c^2(a+b)}{a^2+ab+b^2} \geq \frac{2(a^2+b^2+c^2)}{a+b+c}$$

$$\Leftrightarrow \frac{2rs^2\left(2Rs^4 - rs^2\left(32R^2+4Rr-r^2\right)+6Rr^2(4R+r)^2\right)}{r^2\left(s^2(4R+r)^2 - s^4 - r(4R+r)^3\right)} \geq 2\left(s^2-8Rr-2r^2\right)$$

$$\Leftrightarrow (2R+r)s^6 - rs^4\left(48R^2+20Rr+2r^2\right) + r^2s^2\left(288R^3+192R^2r+42Rr^2+3r^3\right)$$

$$-2r^3(4R+r)^4 \overset{(\bullet)}{\geq} 0 \text{ and } \because (2R+r)\left(s^2-16Rr+5r^2\right)^3$$

$$+r\left(48R^2-2Rr-17r^2\right)\left(s^2-16Rr+5r^2\right)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (\bullet),$$

$$\text{it suffices to prove : LHS of } (\bullet) \geq (2R+r)\left(s^2-16Rr+5r^2\right)^3 + r\left(48R^2-2Rr-17r^2\right)\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow \left(144R^3-80R^2r-76Rr^2+49r^3\right)s^2 \overset{(\bullet\bullet)}{\geq} r\left(2304R^4-2048R^3r-600R^2r^2+876Rr^3-149r^4\right)$$

$$\text{Now, LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq} \left(144R^3-80R^2r-76Rr^2+49r^3\right)\left(16Rr-5r^2\right)$$

$$\overset{?}{\geq} r\left(2304R^4-2048R^3r-600R^2r^2+876Rr^3-149r^4\right)$$

$$\Leftrightarrow 2t^3-9t^2+12t-4 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (2t-1)(t-2)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \frac{\left(\sum_{\text{cyc}} a\right)\left(\sum_{\text{cyc}}\left(a^2(b+c)(c^2+ca+a^2)(a^2+ab+b^2)\right)\right)}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)} \geq 2\sum_{\text{ccy}} a^2$$

$$\Rightarrow \frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2} \geq \frac{2(a^2+b^2+c^2)}{a+b+c}$$

$$\therefore \text{combining both cases, } \frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2} \geq \frac{2(a^2+b^2+c^2)}{a+b+c}\ \forall a,b,c \geq 0 \mid ab+bc+ca>0,$$

$''='' $ iff $(a=0, b=c\neq 0)$ or $(b=0, c=a\neq 0)$ or $(c=0, a=b\neq 0)$ or $(a=b=c\neq 0)$ (QED)

Solution 2 by Nguyen Van Canh-Vietnam

WLOG, we assume that $a+b+c=1$. **Let** $q=ab+bc+ca\leq\frac{(a+b+c)^2}{3}=\frac{1}{3}, r=abc$.

We have:

$$\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2} \geq 2(a^2+b^2+c^2);$$

$$\Leftrightarrow \frac{\sum[a^2(b+c)(c^2+ca+a^2)(a^2+ab+b^2)]}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)} \geq 2(a^2+b^2+c^2);$$

$$\Leftrightarrow \frac{\sum ab(a^5+b^5)+\sum a^2b^2(a^3+b^3)+\sum a^3b^3(a+b)+2abc\sum a^4+2abc\sum ab(a^2+b^2)+4a^2b^2c^2\sum a+2abc\sum a^2b^2}{\sum a^2b^2(a^2+b^2)+\sum a^3b^3+abc\sum a^3+2abc\sum ab(a+b)+3(abc)^2} \geq 2(a^2+b^2+c^2);$$

$$\Leftrightarrow \frac{\sum a^5\sum ab+abc\sum a^4+\sum a^2b^2\sum a^3+3(abc)^2\sum a+\sum a^3b^3\sum a+abc\sum a^2b^2+2abc\sum ab(a^2+b^2)}{\sum a^2b^2\sum a^2+\sum a^3b^3+abc\sum a^3+2abc\sum ab(a+b)} \geq 2\left(a^2+b^2+c^2\right);$$

$$\Leftrightarrow \frac{q(1-5q+5q^2+5r-5qr)+r(1-4q+2q^2+4r)+(q^2-2r)(1-3q+3r)+3r^2+q^3-3qr+3r^2+r(q^2-2r)+2r(q-2q^2-r)}{(q^2-2r)(1-2q)+q^3-3qr+3r^2+r(1-3q+3r)+2r(q-3r)} \geq 2(1-2q):$$

$$\Leftrightarrow \frac{3q^3-4q^2+q+(6q-3q^2-1)r}{q^2-q^3-r} \geq 2(1-2q);$$

$$\Leftrightarrow 3q^3-4q^2+q+(6q-3q^2-1)r \geq 2(1-2q)(q^2-q^3-r);$$

$$\Leftrightarrow (1+2q-3q^2)r-4q^4+9q^3-6q^2+q\ge 0;$$
$$\Leftrightarrow (1-q)(1+3q)r-4q^4+9q^3-6q^2+q\ge 0 \quad (*)$$

By Schur's Inequality we have: $r\ge \max\left\{0,\frac{4q-1}{9}\right\}$.

- If $0<q\le\frac{1}{4}$ then $r\ge 0$, we have:

$$(1-q)(1+3q)r-4q^4+9q^3-6q^2+q\ge -4q^4+9q^3-6q^2+q$$
$$=q(1-4q)(q-1)^2\ge 0 \ \text{(true)} \Rightarrow (*) \text{ true.}$$

- If $\frac{1}{4}\le q\le\frac{1}{3}$ then $r\ge\frac{4q-1}{9}$, we have:

$$(1-q)(1+3q)r-4q^4+9q^3-6q^2+q$$
$$\ge\frac{(1-q)(1+3q)(4q-1)}{9}-4q^4+9q^3-6q^2+q$$
$$=-4q^4+\frac{23q^3}{3}-\frac{43q^2}{9}+\frac{11q}{9}-\frac{1}{9}$$
$$=\frac{1}{9}(1-q)(4q-1)(3q-1)^2\ge 0 \ \text{(true)}\Rightarrow (*) \text{ true.}$$

Proved. Equality $\Leftrightarrow a=b=c$ or $a=0, b=c\neq 0$ or $b=0, a=c\neq 0$ or $c=0, a=b\neq 0$.

1377. Let $a,b,c\ge 0$ and $ab+bc+ca>0$. Prove that:

$$\frac{1}{\left(a+\sqrt{ab}+b\right)^2}+\frac{1}{\left(c+\sqrt{cb}+b\right)^2}+\frac{1}{\left(a+\sqrt{ac}+c\right)^2}\ge\frac{1}{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Nguyen Van Canh-Vietnam

By AM-GM we have: $\sqrt{ab}\le\frac{a+b}{2}$

$$\Rightarrow \frac{1}{\left(a+\sqrt{ab}+b\right)^2}\ge\frac{1}{\left(a+\frac{a+b}{2}+b\right)^2}=\frac{4}{9(a+b)^2} \text{ (and analogs)}$$

$$\Rightarrow \frac{1}{\left(a+\sqrt{ab}+b\right)^2}+\frac{1}{\left(c+\sqrt{cb}+b\right)^2}+\frac{1}{\left(a+\sqrt{ac}+c\right)^2}$$
$$\ge\frac{4}{9}\left[\frac{1}{(a+b)^2}+\frac{1}{(c+b)^2}+\frac{1}{(a+c)^2}\right];$$

So that we need to prove:

$$\frac{4}{9}\left[\frac{1}{(a+b)^2}+\frac{1}{(c+b)^2}+\frac{1}{(a+c)^2}\right]\ge\frac{1}{ab+bc+ca};$$
$$\Leftrightarrow (ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(c+b)^2}+\frac{1}{(a+c)^2}\right]\ge\frac{9}{4}.$$

Which is true because this is Iran Inequality 1996. Proved.

1378. If $a,b,c>0, ab+bc+ca=abc$ and $3\le\lambda\le4$, then :

$$a+b+c\ge\lambda\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-1\right)+9$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=r^2s\rightarrow(2)$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\rightarrow(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(4)$$

$$\text{We have}: a+b+c-\lambda\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-1\right)-9$$

$$=\frac{(\sum_{cyc}a)(\sum_{cyc}ab)}{abc}-\lambda\left(\frac{\sum_{cyc}a^2-\sum_{cyc}ab}{\sum_{cyc}ab}\right)-9\left(\because\sum_{cyc}ab=abc\right)$$

$$\overset{\text{via (1),(2),(3) and (4)}}{=}\frac{s(4Rr+r^2)}{r^2s}-\lambda\left(\frac{s^2-8Rr-2r^2-4Rr-r^2}{4Rr+r^2}\right)-9$$

$$=\frac{4R+r}{r}-\lambda\left(\frac{s^2-12Rr-3r^2}{4Rr+r^2}\right)-9\overset{0\le\lambda\le4}{\ge}\frac{4R+r}{r}-4\left(\frac{s^2-12Rr-3r^2}{4Rr+r^2}\right)-9$$

$$\left(\because s^2-12Rr-3r^2=s^2-16Rr+5r^2+4r(R-2r)\overset{\text{Gerretsen and Euler}}{\ge}0\right)$$

$$=\frac{(4R+r)^2-4(s^2-12Rr-3r^2)-36Rr-9r^2}{4Rr+r^2}\overset{\text{Gerretsen}}{\ge}$$

$$\frac{(4R+r)^2-4(4R^2+4Rr+3r^2-12Rr-3r^2)-36Rr-9r^2}{4Rr+r^2}=\frac{4r(R-2r)}{4Rr+r^2}\overset{\text{Euler}}{\ge}0$$

$$\therefore a+b+c\ge\lambda\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-1\right)+9\ \forall\ a,b,c>0,\sum_{cyc}ab=abc \text{ and } 3\le\lambda\le4,$$

$''=''$ iff $a=b=c=3$ (QED)

1379. $If\ a,b,c>0\Rightarrow$

$$\frac{(a+b+c)^2}{\ln\left((a+1)^{e^a-1}.(b+1)^{e^b-1}.(c+1)^{e^c-1}\right)}+2\sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right)<9$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall first prove that}:(e^x-1).\ln(x+1)>x^2\ \forall\,x>0$$

$$\because e^x-1\ge x+\frac{x^2}{2}\ \forall\,x>0\ \therefore (e^x-1).\ln(x+1)\ge\left(x+\frac{x^2}{2}\right).\ln(x+1)\overset{?}{>}x^2$$

$$\Leftrightarrow \ln(x+1)\underset{(i)}{\overset{?}{>}}\frac{x}{1+\frac{x}{2}}$$

$$\text{Let } f(x)=\ln(x+1)-\frac{x}{1+\frac{x}{2}}\ \forall\,x\ge 0 \text{ and then}: f'(x)=\frac{x^2}{(x+1)(x+2)^2}\ge 0$$

$$\Rightarrow f(x)\text{ is }\uparrow\text{ on }[0,\infty)\Rightarrow\forall\,x\ge0, f(x)\ge f(0)=0\ \therefore\ \forall\,x>0, f(x)>f(0)$$

$$\Rightarrow (i)\text{ is true }\therefore \boxed{(e^x-1).\ln(x+1)>x^2\ \forall\,x>0}\rightarrow(1)$$

$$\text{We shall now prove that}:\sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right)\le 3$$

$$\Leftrightarrow 2\sum_{cyc}x^2-\sum_{cyc}x^3\le 3\ \left(x=\sqrt{\frac{a}{b}},y=\sqrt{\frac{b}{c}},z=\sqrt{\frac{c}{a}}\right)$$

$$\Leftrightarrow \sum_{cyc}x^3+3xyz\ge\left(2\sum_{cyc}x^2\right).\sqrt[3]{xyz}\ (\because xyz=1)$$

$$\Leftrightarrow\left(\sum_{cyc}x^3+3xyz\right)^3\overset{(*)}{\ge}8xyz\left(\sum_{cyc}x^2\right)^3$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ *form* sides of *a* triangle with semiperimeter, circumradius *and* inradius $=s,R,r$ (*say*) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\rightarrow(1)\Rightarrow x=s-X, y=s-Y,$

$z=s-Z$ *and such substitutions* $\Rightarrow\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)$

$$\Rightarrow\sum_{cyc}xy=4Rr+r^2\rightarrow(2),\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via (1) and (2)}}{=}$$

$$s^2-(4Rr+r^2)\Rightarrow\sum_{cyc}x^2=s^2-8Rr-2r^2\rightarrow(3)\text{ and}$$

$$\sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3(x+y)(y+z)(z+x) \overset{via\ (1)}{=} s^3 - 3XYZ = s^3 - 12Rrs$$

$$\Rightarrow \sum_{cyc} x^3 = s(s^2-12Rr) \rightarrow (4)\ and\ xyz = (s-X)(s-Y)(s-Z) = r^2s \rightarrow (5)$$

$$\therefore via\ (3),(4),(5),(*) \Leftrightarrow \left(s(s^2-12Rr)+3r^2s\right)^3 \ge 8r^2s(s^2-8Rr-2r^2)^3$$

$$\Leftrightarrow s^8 - (36Rr-r^2)s^6 + r^2(432R^2-24Rr+75r^2)s^4$$

$$-r^3(1728R^3+240R^2r+1092Rr^2+69r^3)s^2+64r^5(4R+r)^3 \overset{(**)}{\ge} 0$$

Now, via Gerretsen, $(s^2-16Rr+5r^2)^4 + (28Rr-19r^2)(s^2-16Rr+5r^2)^3$

$+r^2(240R^2-396Rr+210r^2)(s^2-16Rr+5r^2)^2 \ge 0 \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**) \ge (s^2-16Rr+5r^2)^4$

$$+(28Rr-19r^2)(s^2-16Rr+5r^2)^3$$

$$+r^2(240R^2-396Rr+210r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (208R^3-660R^2r+792Rr^2-311r^3)s^2 \overset{(***)}{\ge}$$

$$r(3072R^4-10112R^3r+12972R^2r^2-6492Rr^3+859r^4)$$

$$\because 208R^3-660R^2r+792Rr^2-311r^3$$

$$= (R-2r)(208R^2-244Rr+304r^2)+297r^3 \overset{Euler}{\ge} 297r^3 > 0$$

$$\therefore \text{LHS of } (***) \overset{Rouche}{\ge}$$

$$(208R^3-660R^2r+792Rr^2-311r^3)\begin{pmatrix} 2R^2+10Rr-r^2 \\ -2(R-2r)\sqrt{R^2-2Rr} \end{pmatrix}$$

$$\overset{?}{\ge} r(3072R^4-10112R^3r+12972R^2r^2-6492Rr^3+859r^4)$$

$$\Leftrightarrow \boxed{\begin{matrix} (R-2r)(208R^4-740R^3r+964R^2r^2-579Rr^3+137r^4) \overset{?}{\underset{(****)}{\ge}} \\ (R-2r)\sqrt{R^2-2Rr}.(208R^3-660R^2r+792Rr^2-311r^3) \end{matrix}}$$

$$\because 208R^4-740R^3r+964R^2r^2-579Rr^3+137r^4$$

$$= (R-2r)(46R^3+162R^2(R-2r)+316Rr^2+53r^3)+243r^3 \overset{Euler}{\ge} 243r^3 > 0$$

$and \because R-2r \overset{Euler}{\ge} 0 \therefore$ in order to prove $(****)$, it suffices to prove :

$$(208R^4-740R^3r+964R^2r^2-579Rr^3+137r^4)^2$$

$$> (R^2-2Rr)(208R^3-660R^2r+792Rr^2-311r^3)^2$$

$$\Leftrightarrow 53248t^7-365568t^6+1037376t^5-1544208t^4+1249120t^3-482592t^2$$

$$+34796t+18769 > 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)\big((t-2)P+987552\big)+335340\Big)+59049 > 0$$

where $P = 30208t^4+23040t^3(t-2)+121920t^2+166256t+414976 \rightarrow$ true

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \boxed{\sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right) \overset{(\bullet)}{\leq} 3}$$

$$\text{Again, } \frac{(a+b+c)^2}{\ln\left((a+1)^{e^a-1}.(b+1)^{e^b-1}.(c+1)^{e^c-1}\right)}$$

$$= \frac{(a+b+c)^2}{(e^a-1).\ln(a+1)+(e^b-1).\ln(b+1)+(e^c-1).\ln(c+1)} \overset{\text{via }(1)}{<} \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

$$\leq 3 \therefore \boxed{\frac{(a+b+c)^2}{\ln\left((a+1)^{e^a-1}.(b+1)^{e^b-1}.(c+1)^{e^c-1}\right)} \overset{(\bullet\bullet)}{<} 3} \therefore (\bullet)+(\bullet\bullet) \Rightarrow$$

$$\frac{(a+b+c)^2}{\ln\left((a+1)^{e^a-1}.(b+1)^{e^b-1}.(c+1)^{e^c-1}\right)} + 2\sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right) < 9$$

$$\forall a,b,c > 0 \ (QED)$$

1380. If $0 < x,y,z < 1, x+y+z = \frac{3}{2}$, then :

$$\sum_{cyc}\frac{\sqrt{1+x^2}}{1-x} \geq 3\sqrt{5}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\sqrt{1+x^2}}{1-x} = \sum_{cyc}\frac{\left(\sqrt[4]{1+x^2}\right)^2}{1-x} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}\sqrt[4]{1+x^2}\right)^2}{3-(x+y+z)} \overset{x+y+z=\frac{3}{2}}{=}$$

$$\frac{2}{3}\cdot\left(\sum_{cyc}\sqrt[4]{1+x^2}\right)^2 \overset{?}{\geq} 3\sqrt{5} \Leftrightarrow \sum_{cyc}\sqrt[4]{1+x^2} \underset{(*)}{\overset{?}{\geq}} \frac{3\sqrt[4]{5}}{\sqrt{2}}$$

$$\text{Now, } f(t) = \sqrt[4]{1+t^2}\ \forall t \in (0,1) \text{ is convex} \because f''(t) = \frac{2-t}{4(1+t^2)^{\frac{7}{4}}} > 0\ (\because 0 < t < 1)$$

$$\therefore \sum_{cyc}\sqrt[4]{1+x^2} \overset{\text{Jensen}}{\geq} 3\sqrt[4]{1+\left(\frac{x+y+z}{3}\right)^2} \overset{x+y+z=\frac{3}{2}}{=} \frac{3\sqrt[4]{5}}{\sqrt{2}} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{\sqrt{1+x^2}}{1-x} \geq 3\sqrt{5}\ \forall x,y,z \in (0,1), '' ='' \text{ iff } x=y=z=\frac{1}{2}\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove a lemma that for all $x \in (0,1)$,

$$\frac{\sqrt{1+x^2}}{1-x} \geq \frac{12x-1}{\sqrt{5}}.$$

The inequality is successively equivalent to

$$5(1+x^2) \geq (1-x)^2(12x-1)^2 \Leftrightarrow 2+13x-94x^2+156x^3-72x^4 \geq 0$$

$$\Leftrightarrow (1-2x)^2[2+3x+18x(1-x)] \geq 0,$$

which is true for all $x \in (0,1)$, with equality iff $x = \frac{1}{2}$.

Using this lemma, we have

$$\frac{\sqrt{1+x^2}}{1-x} + \frac{\sqrt{1+y^2}}{1-y} + \frac{\sqrt{1+z^2}}{1-z} \geq \frac{12(x+y+z)-3}{\sqrt{5}} = 3\sqrt{5},$$

as desired. Equality holds iff $x = y = z = \frac{1}{2}$.

1381. If $0 < x, y, z < 1, x+y+z = \frac{3}{2}$ and $0 \leq \lambda \leq 1$, then :

$$\sum_{cyc} \frac{\sqrt{\lambda + x^2}}{1-x} \geq 3.\sqrt{4\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{4\lambda+1} - \frac{\sqrt{\lambda+x^2}}{1-x} = \frac{4\lambda+1-\frac{\lambda+x^2}{(1-x)^2}}{\sqrt{4\lambda+1}+\frac{\sqrt{\lambda+x^2}}{1-x}} = \frac{\frac{(4\lambda+1)(1+x^2-2x)-\lambda-x^2}{(1-x)^2}}{\sqrt{4\lambda+1}+\frac{\sqrt{\lambda+x^2}}{1-x}}$$

$$= \frac{1-2x+\lambda(4x^2-8x+3)}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}} = \frac{1-2x-\lambda(1-2x)(2x-3)}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}}$$

$$= \frac{(1-2x)(1-\lambda(2x-3))}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}} \therefore \boxed{\sqrt{4\lambda+1} - \frac{\sqrt{\lambda+x^2}}{1-x} - \frac{(1-2x)(4\lambda+2)}{\sqrt{4\lambda+1}}}$$

$$= (1-2x)\left(\frac{1-\lambda(2x-3)}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}} - \frac{4\lambda+2}{\sqrt{4\lambda+1}}\right)$$

$$= -(1-2x).\frac{(4\lambda+2).\sqrt{4\lambda+1}.(1-2x+x^2)+(4\lambda+2)(1-x).\sqrt{\lambda+x^2} - \sqrt{4\lambda+1}+\lambda(2x-3).\sqrt{4\lambda+1}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)\left(\begin{matrix}(4\lambda+2).\sqrt{4\lambda+1}.(1-2x)+(4\lambda+2).\sqrt{4\lambda+1}.x^2+ \\ (4\lambda+2)\left(\frac{1}{2}-x+\frac{1}{2}\right).\sqrt{\lambda+x^2}-\sqrt{4\lambda+1}+\lambda(2x-1).\sqrt{4\lambda+1}-2\lambda.\sqrt{4\lambda+1}\end{matrix}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)\left(\begin{matrix}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+ \\ (2\lambda+1)\left(2\left(x^2-\frac{1}{4}+\frac{1}{4}\right).\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-\sqrt{4\lambda+1}\right)\end{matrix}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$=-\frac{(1-2x)\begin{pmatrix}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+\\(2\lambda+1)\left(-2\left(\frac{1}{4}-x^2\right).\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-\frac{1}{2}\sqrt{4\lambda+1}\right)\end{pmatrix}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$=-\frac{(1-2x)\begin{pmatrix}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+\\(2\lambda+1)\left(-2\left(\frac{1}{4}-x^2\right).\sqrt{4\lambda+1}-\frac{\frac{1}{4}(4\lambda+1)-\lambda-x^2}{\frac{1}{2}\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}}\right)\end{pmatrix}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$=-\frac{(1-2x)\begin{pmatrix}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+\\(2\lambda+1)(1-2x)\left(-\left(\frac{1}{2}+x\right).\sqrt{4\lambda+1}-\frac{\frac{1}{2}+x}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)\end{pmatrix}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}=$$

$$-\frac{(1-2x)^2(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}-\left(\frac{1}{2}+x\right)\left(\sqrt{4\lambda+1}+\frac{1}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$\le-\frac{(1-2x)^2(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+1}-\left(\frac{1}{2}+x\right).\frac{4\lambda+2+2\sqrt{4\lambda+1}.\sqrt{\lambda+x^2}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$\boxed{\le}$$

$$-\frac{(1-2x)^2(2\lambda+1)\left(\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-\left(\frac{1}{2}+x\right).\frac{4\lambda+2+2\sqrt{4\lambda+1}.\sqrt{\lambda+x^2}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)-\frac{2\lambda}{\sqrt{4\lambda+1}}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$=-\frac{(1-2x)^2(2\lambda+1)\left(\frac{8\lambda+1+2x^2-4x\lambda-2x+(4-2x).\sqrt{(4\lambda+1)(\lambda+x^2)}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}-\frac{2\lambda}{\sqrt{4\lambda+1}}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$=-\frac{(1-2x)^2(2\lambda+1)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}*$$

$$\left(\frac{4\lambda(2-x)+\left(1+2x^2-2x\right)+(4-2x).\sqrt{(4\lambda+1)(\lambda+x^2)}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}-\frac{2\lambda}{\sqrt{4\lambda+1}}\right)$$

$$= -\frac{(1-2x)^2(2\lambda+1)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}*$$

$$\left(\frac{4\lambda.\sqrt{4\lambda+1}.(2-x)+\left(1+2x^2-2x\right).\sqrt{4\lambda+1}+(4-2x)(4\lambda+1).\sqrt{\lambda+x^2}-2\lambda.\sqrt{4\lambda+1}-4\lambda.\sqrt{\lambda+x^2}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}\right)}\right)$$

$$= -\frac{(1-2x)^2(2\lambda+1)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}*$$

$$\left(\frac{2\lambda.\sqrt{4\lambda+1}.(3-2x)+\sqrt{\lambda+x^2}.(4\lambda(3-2x)+4-2x)+(1-x)^2.\sqrt{4\lambda+1}+x^2.\sqrt{4\lambda+1}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}\right)}\right)$$

$$\boxed{\le 0} \because 0<x<1 \text{ and } \lambda\ge 0 \Rightarrow 2\lambda.\sqrt{4\lambda+1}.(3-2x)+$$
$$\sqrt{\lambda+x^2}.(4\lambda(3-2x)+4-2x)+(1-x)^2.\sqrt{4\lambda+1}+x^2.\sqrt{4\lambda+1}>0$$

$$\therefore \sqrt{4\lambda+1}-\frac{\sqrt{\lambda+x^2}}{1-x}-\frac{(1-2x)(4\lambda+2)}{\sqrt{4\lambda+1}}\le 0\ \forall\, x\in(0,1) \text{ and } \forall\,\lambda\ge 0$$

$$\Rightarrow \boxed{\boxed{\frac{\sqrt{\lambda+x^2}}{1-x}\ge\sqrt{4\lambda+1}-\frac{(1-2x)(4\lambda+2)}{\sqrt{4\lambda+1}}\ \forall\, x\in(0,1) \text{ and } \forall\,\lambda\ge 0}} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\frac{\sqrt{\lambda+x^2}}{1-x}\ge 3.\sqrt{4\lambda+1}-\frac{4\lambda+2}{\sqrt{4\lambda+1}}.\left(3-2\sum_{cyc}x\right)$$

$$\overset{x+y+z=\frac{3}{2}}{=} 3.\sqrt{4\lambda+1}\ \forall\, x,y,z\in(0,1) \text{ and } \forall\,\lambda\in[0,1], ''='' \text{ iff } x=y=z=\frac{1}{2} \text{ (QED)}$$

1382. If $a,b,c>0, n\in\mathbb{N}$ then:

$$\sum_{cyc}\frac{a^{n+1}}{b^n+c^n}\ge\frac{a+b+c}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

For $n=0$: $\frac{a+b+c}{2}=\frac{a+b+c}{2}$.

For $n\in\mathbf{N}^*$:

$\left(a^{n+1},b^{n+1},c^{n+1}\right)$, $\left(\frac{1}{b^n+c^n},\frac{1}{c^n+a^n},\frac{1}{a^n+b^n}\right)$ are same ordered.

By Chebyshev :

$$LHS=\sum\frac{a^{n+1}}{b^n+c^n}\overset{Chebyshev}{\ge}\frac{1}{3}\sum a^{n+1}\sum\frac{1}{b^n+c^n}\overset{CS}{\ge}\frac{1}{3}\sum a^{n+1}\frac{9}{\sum\left(b^n+c^n\right)}=\frac{1}{3}\sum a^{n+1}\frac{9}{2\sum a^n}=$$

$$=\frac{3\sum a^{n+1}}{2\sum a^n}\overset{(1)}{\geq}\frac{\sum a}{2}=RHS,$$

$$\textbf{(1)}\Leftrightarrow\frac{3\sum a^{n+1}}{2\sum a^n}\geq\frac{\sum a}{2}\Leftrightarrow 3\sum a^{n+1}\geq\sum a^n\sum a.$$

Equality holds for $a=b=c$.

Solution 2 by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a^{n+1}}{b^n+c^n}\overset{LEHMER}{\geq}\sum_{cyc}\frac{a^n}{b^{n-1}+c^{n-1}}\overset{LEHMER}{\geq}\sum_{cyc}\frac{a^{n-1}}{b^{n-2}+c^{n-2}}\overset{LEHMER}{\geq}$$

$$\geq\sum_{cyc}\frac{a^2}{b+c}\overset{BERGSTROM}{\geq}\frac{(a+b+c)^2}{b+c+c+a+a+b}=\frac{a+b+c}{2}$$

Equality holds for $a=b=c$.

1383. **If $a,b,c,\lambda>0$ and $ab+bc+ca=\lambda$, then :**

$$\frac{\sum_{cyc}\sqrt{a^2+\lambda}}{\sum_{cyc}\sqrt{ab}}\geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}\sqrt{a^2+\lambda}}{\sum_{cyc}\sqrt{ab}}\geq 2\Leftrightarrow\frac{\sum_{cyc}\sqrt{a^2+ab+bc+ca}}{\sum_{cyc}\sqrt{ab}}\geq 2$$

$$\Leftrightarrow\boxed{\sum_{cyc}\sqrt{(a+b)(c+a)}\overset{(*)}{\geq}2\sum_{cyc}\sqrt{ab}}$$

Assigning $b+c=x',c+a=y',a+b=z'\Rightarrow x'+y'-z'=2c>0,y'+z'-x'=2a>0$ and $z'+x'-y'=2b>0\Rightarrow x'+y'>z',y'+z'>x',z'+x'>y'$ $\Rightarrow x',y',z'$ form sides of a triangle $\Rightarrow\sqrt{b+c}=\sqrt{x'}=x,\sqrt{c+a}=\sqrt{y'}=y$, $\sqrt{a+b}=\sqrt{z'}=z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

$$\text{Now, } b+c=x^2,c+a=y^2,a+b=z^2\Rightarrow\sum_{cyc}a=\frac{x^2+y^2+z^2}{2}$$

$$\Rightarrow a=\frac{y^2+z^2-x^2}{2},b=\frac{z^2+x^2-y^2}{2},c=\frac{x^2+y^2-z^2}{2}$$

$$\therefore\text{ using such transformations, }\sum_{cyc}\sqrt{(a+b)(c+a)}\geq 2\sum_{cyc}\sqrt{ab}$$

$$\Leftrightarrow \sum_{cyc} yz \geq 2\sum_{cyc}\sqrt{\frac{y^2+z^2-x^2}{2}.\frac{z^2+x^2-y^2}{2}}$$

$$\Leftrightarrow \boxed{\begin{array}{l}\sum_{cyc} x^2y^2 + 2xyz\sum_{cyc} x \overset{(**)}{\geq} \sum_{cyc}(y^2+z^2-x^2)(z^2+x^2-y^2) \\ +2\sum_{cyc}\left((y^2+z^2-x^2).\sqrt{(z^2+x^2-y^2)(x^2+y^2-z^2)}\right)\end{array}}$$

$$\text{Now, RHS of } (**) \overset{A-G}{\leq} 2\sum_{cyc}\left((y^2+z^2-x^2).\frac{(z^2+x^2-y^2)+(x^2+y^2-z^2)}{2}\right)$$

$$= 2\sum_{cyc} x^2(y^2+z^2-x^2) = 4\sum_{cyc} x^2y^2 - 2\sum_{cyc} x^4$$

$$\therefore \text{LHS of } (**) \leq -\sum_{cyc} x^4 + 2\sum_{cyc} x^2y^2 + 4\sum_{cyc} x^2y^2 - 2\sum_{cyc} x^4 \overset{?}{\leq}$$

$$\sum_{cyc} x^2y^2 + 2xyz\sum_{cyc} x \Leftrightarrow 3\left(2\sum_{cyc} x^2y^2 - 16r^2s^2\right) - 5\sum_{cyc} x^2y^2 + 2.4Rrs.2s \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(\sum_{cyc} xy\right)^2 - 16Rrs^2 - 48r^2s^2 + 16Rrs^2 \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc} xy \overset{?}{\geq} 4\sqrt{3}rs$$

$$\rightarrow \text{true via Gordon} \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{\sum_{cyc}\sqrt{a^2+\lambda}}{\sum_{cyc}\sqrt{ab}} \geq 2\ \forall\, a,b,c,\lambda > 0$$

$$\text{and } ab+bc+ca = \lambda, '' ='' \text{ iff } a = b = c = \sqrt{\frac{\lambda}{3}}\ (\text{QED})$$

1384. If $x, y, z > 0$ such that : $x + y + z = xyz$ and $\frac{3}{2} \leq \lambda \leq 2$, then :

$$x^2 + y^2 + z^2 + 9(\lambda - 1) \geq \lambda(xy + yz + zx)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(i)}{=} s$$

$$\Rightarrow x = s - a, y = s - b, z = s - c \Rightarrow xyz \overset{(ii)}{=} r^2s$$

$$\text{Via such substitutions, } \sum_{cyc} xy = \sum_{cyc}(s-a)(s-b) = 4Rr + r^2$$

$$\Rightarrow \sum_{cyc} xy \overset{(1)}{=} 4Rr + r^2 \Rightarrow \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy = s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 \overset{(2)}{=} s^2 - 8Rr - 2r^2$$

$$\Leftrightarrow x^2 + y^2 + z^2 + 9(\lambda - 1) \geq \lambda(xy + yz + zx)$$

$$\Leftrightarrow \sum_{cyc} x^2 . \frac{\sum_{cyc} x}{xyz} - 9 \geq \lambda\left(\sum_{cyc} xy . \frac{\sum_{cyc} x}{xyz} - 9\right) \left(\because 1 = \frac{x + y + z}{xyz}\right)$$

$$\Leftrightarrow \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) - 9xyz \overset{(*)}{\geq} \lambda\left(\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right) - 9xyz\right)$$

$$\text{Now, } \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) - 9xyz - \lambda\left(\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right) - 9xyz\right)$$

$$\geq s(s^2 - 8Rr - 2r^2) - 9r^2s - 2(s(4Rr + r^2) - 9r^2s) \left(\begin{array}{c}\because 0 < \lambda \leq 2 \text{ and} \\ \text{via (i), (ii), (1) and (2)}\end{array}\right)$$

$$= s(s^2 - 16Rr + 5r^2) \overset{\text{Gerretsen}}{\geq} 0 \Rightarrow (*) \text{ is true } \therefore x^2 + y^2 + z^2 + 9(\lambda - 1)$$

$$\geq \lambda(xy + yz + zx)\ \forall\, x, y, z > 0 \,\middle|\, x + y + z = xyz \text{ and } \frac{3}{2} \leq \lambda \leq 2,$$

$$'' ='' \text{ iff } x = y = z = \sqrt{3} \text{ (QED)}$$

1385. If $a, b \geq 0$, prove that

$$\sqrt[3]{e^{a^2+ab+b^2}}^{\,8} + \sqrt[3]{e^{a^2+6ab+b^2}} \geq e^{2a^2+2b^2} + e^{(a+b)^2} + e^{ab}\left(e^{3ab} - \sqrt[3]{e^{ab}}\right)$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since the sequence $\left(\frac{8(a^2 + ab + b^2)}{3}, \frac{a^2 + 6ab + b^2}{3}, \frac{4ab}{3}\right)$ majorizes the sequence $(2a^2 + 2b^2, (a + b)^2, 4ab)$ and since the function $f(x) = e^x, x \geq 0$ is convex, then by Karamata's inequality, we have

$$f\left(\frac{8(a^2 + ab + b^2)}{3}\right) + f\left(\frac{a^2 + 6ab + b^2}{3}\right) + f\left(\frac{4ab}{3}\right)$$

$$\geq f(2a^2 + 2b^2) + f((a + b)^2) + f(4ab),$$

which is equivalent to

$$\sqrt[3]{e^{a^2+ab+b^2}}^{\,8} + \sqrt[3]{e^{a^2+6ab+b^2}} \geq e^{2a^2+2b^2} + e^{(a+b)^2} + e^{ab}\left(e^{3ab} - \sqrt[3]{e^{ab}}\right),$$

as desired. Equality holds iff $a = b = 0$.

1386. Let $a,b,c\ge 0$ such that $ab+bc+ca=3$. Prove that

$$\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}\ge 3\sqrt{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $0\le a\le b\le c$. Since $3=ab+bc+ca\ge 3a^2$, then $0\le a\le 1$.

We have $a^2+3=(a+b)(a+c)\le\left(\frac{2a+b+c}{2}\right)^2$, then $b+c\ge 2\left(\sqrt{a^2+3}-a\right)$.

Let $t:=\sqrt{a^2+3}-a\in[1,\sqrt{3}]$. We have $b+c\ge 2t$, $a=\frac{3-t^2}{2t}$, and

$$\left(\sqrt{b+1}+\sqrt{c+1}\right)^2=b+c+2+2\sqrt{b+c+bc+1}=b+c+2+2\sqrt{(b+c)(1-a)+4}$$

$$\ge 2t+2+2\sqrt{2t\left(1-\frac{3-t^2}{2t}\right)+4}=4(t+1).$$

Then

$$\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}\ge\sqrt{a+1}+2\sqrt{t+1}=\sqrt{\frac{3-t^2}{2t}+1}+2\sqrt{t+1}\overset{?}{\ge}3\sqrt{2}$$

$$\Leftrightarrow 4\sqrt{\left(\frac{3-t^2}{2t}+1\right)(t+1)}\ge 13-4t-\frac{3-t^2}{2t}.$$

If $13\le 4t+\frac{3-t^2}{2t}$, the last inequality is true.

Otherwise, after squaring, the last inequality is equivalent to

$$-1+28t-62t^2+44t^3-9t^4\ge 0\Leftrightarrow (t-1)^2[t(25-9t)+t-1]\ge 0,$$

which is true for all $t\in[1,\sqrt{3}]$. So the proof is complete.

Equality holds iff $a=b=c=1$.

1387. If $x,y,z>0, x+y+z=1$ then:

$$\sqrt{xy-2z+6}+\sqrt{yz-2x+6}+\sqrt{zx-2y+6}\le 7$$

Proposed by Samed Ahmedov-Azerbaijan

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}xy\overset{GM-QM}{\le}\sum_{cyc}\frac{x^2+y^2}{2}=\sum_{cyc}x^2=(x+y+z)^2-2\sum_{cyc}xy=$$

$$=1-2\sum_{cyc}xy\Rightarrow 3\sum_{cyc}xy\le 1\Rightarrow$$

$$\sum_{cyc}xy\le\frac{1}{3}\quad (1)$$

$$\sum_{cyc}\sqrt{xy-2z+6}\overset{CBS}{\leq}\sqrt{3\sum_{cyc}(xy-2z+6)}\leq$$

$$\leq\sqrt{3\sum_{cyc}xy-6\sum_{cyc}x+54}\overset{(1)}{\leq}\sqrt{3\cdot\frac{1}{3}-6\cdot 1+54}=\sqrt{49}=7$$

Equality holds for $x=y=z=\frac{1}{3}$.

1388. If $a,b,c>0$ such that $abc=1$ and $n,m,t\in\mathbb{N}$, then prove that

$$\frac{a^t(b^{n+m}+c^{n+m})}{a^{n+m}+b^nc^m}+\frac{b^t(c^{n+m}+a^{n+m})}{b^{n+m}+c^na^m}+\frac{c^t(a^{n+m}+b^{n+m})}{c^{n+m}+a^nb^m}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solutions 1,2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 :

By AM − GM inequality, we have

$$\sum_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{a^{n+m}+b^nc^m}\geq 3\sqrt[3]{\frac{(abc)^t(a^{n+m}+b^{n+m})(b^{n+m}+c^{n+m})(c^{n+m}+a^{n+m})}{(a^{n+m}+b^nc^m)(b^{n+m}+c^na^m)(c^{n+m}+a^nb^m)}}\overset{?}{\geq}3$$

$$\Leftrightarrow (a^{n+m}+b^{n+m})(b^{n+m}+c^{n+m})(c^{n+m}+a^{n+m})$$
$$\geq (a^{n+m}+b^nc^m)(b^{n+m}+c^na^m)(c^{n+m}+a^nb^m)$$
$$\Leftrightarrow \sum_{cyc}a^{2(n+m)}(b^{n+m}+c^{n+m})\geq\sum_{cyc}a^{2n+m}b^{n+2m}+\sum_{cyc}a^{2(n+m)}b^mc^n\qquad(*)$$

By AM − GM inequality, we have $a^nb^m\leq\frac{n}{n+m}.a^{n+m}+\frac{m}{n+m}.b^{n+m}$ (and analogs), then

$$RHS_{(*)}\leq\sum_{cyc}a^{n+m}b^{n+m}\left(\frac{n}{n+m}.a^{n+m}+\frac{m}{n+m}.b^{n+m}\right)$$
$$+\sum_{cyc}a^{2(n+m)}\left(\frac{n}{n+m}.c^{n+m}+\frac{m}{n+m}.b^{n+m}\right)=\sum_{cyc}a^{2(n+m)}(b^{n+m}+c^{n+m})=LHS_{(*)},$$

which completes the proof. Equality holds iff $a=b=c=1$.

Solution 2 :

By Hölder's inequality, we have

$$a^{n+m}+b^nc^m\leq\sqrt[n+m]{(a^{n+m}+b^{n+m})^n(a^{n+m}+c^{n+m})^m}\ \text{(and analogs)}$$

Then

$$\sum_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{a^{n+m}+b^nc^m}\geq\sum_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{\sqrt[n+m]{(a^{n+m}+b^{n+m})^n(a^{n+m}+c^{n+m})^m}}$$
$$\overset{AM-GM}{\geq}3\sqrt[3]{\prod_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{\sqrt[n+m]{(a^{n+m}+b^{n+m})^n(a^{n+m}+c^{n+m})^m}}}=3,$$

as desired. Equality holds iff $a=b=c=1$.

1389. Let $a,b,c\geq 0$ such that $ab+bc+ca=3$. Prove that

$$\frac{1}{\sqrt{5a+bc}}+\frac{1}{\sqrt{5b+ca}}+\frac{1}{\sqrt{5c+ab}}\geq\frac{\sqrt{6}}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } p:=a+b+c,\ \ q:=ab+bc+ca=3,\ \ r:=abc\overset{AM-GM}{\leq}1.\ \text{By Hölder's inequality, we have}$$

$$\left(\sum_{cyc}\frac{1}{\sqrt{5a+bc}}\right)^2\left(\sum_{cyc}(5a+bc)(b+c)^3(2a+b+c)^3\right)\geq\left(\sum_{cyc}(b+c)(2a+b+c)\right)^3=8(p^2+3)^3.$$

So it suffices to prove that

$$16(p^2+3)^3\geq 3\sum_{cyc}(5a+bc)(b+c)^3(a+p)^3,$$

which, after expanding and simplifying becomes,

$$f(r)=7p^6-45p^5+144p^4-135p^3+432p^2-2835p+432+3(p^5+5p^4+12p^3-90p^2+45p+315)r-3(4p^2-35p+15)r^2\geq 0.$$

We have $\quad f'(r)=3[p^5+5p^4+12p^3-90p^2+45p+315-2(4p^2-35p+15)r]$.

▪If $4p^2-35p+15\leq 0$, by using AM − GM inequality, we have

$$f'(r)\geq 3(p^5+5p^4+12p^3+45p+315-90p^2)\geq 3\left(4\sqrt[4]{5p^4.12p^3.45p.315}-90p^2\right)>0.$$

▪If $4p^2-35p+15\geq 0$, since $r\leq 1$ and by using AM − GM inequality, we have

$$f'(r)\geq 3(p^5+5p^4+12p^3-98p^2+115p+285)\geq 3\left(4\sqrt[4]{5p^4.12p^3.115p.285}-98p^2\right)>0,$$

then f is increasing. And since $p\geq\sqrt{3q}=3$ then we have two cases :

▪If $p\geq 2\sqrt{3}$, we have

$$f(r)\geq f(0)=(p-2\sqrt{3})[(7p+14\sqrt{3}-45)p^4+(228-90\sqrt{3})p^3+(456\sqrt{3}-675)p^2+(3168-1350\sqrt{3})p+6336\sqrt{3}-10935]+54(712-405\sqrt{3})\geq 0.$$

▪If $3\leq p\leq 2\sqrt{3}$, by Schur's inequality, we have $r\geq\frac{p(4q-p^2)}{9}=r'$, then $f(r)\geq f(r')$ with

$$f(r')=\frac{1}{27}(p-3)\{(2\sqrt{3}-p)[13p^6+(49+26\sqrt{3})p^5+(98\sqrt{3}+33)p^4+(924+66\sqrt{3})p^3+(1848\sqrt{3}-3159)p^2+(11583-6318\sqrt{3})p+23166\sqrt{3}-50787]+101574\sqrt{3}-142884\}\geq 0.$$

So the proof is complete. Equality holds iff $a=b=c=1$.

1390. **If $a,b,c\geq 0$ with $ab+bc+ca\neq 0$, then prove that :**

$$\sum_{cyc}\frac{a^3}{b^2-bc+c^2}\geq a+b+c+AB\sum_{cyc}(a-b)^2(a+b-c)^2((a-b)^2+ab),$$

$$\text{where } A=\frac{2(a^3+b^3+c^3)}{2(a^2+b^2+c^2)+(a-b)^2+(b-c)^2+(c-a)^2};$$

$$B=\frac{(a+b)(b+c)(c+a)}{(a^3+b^3)(b^3+c^3)(c^3+a^3)}$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a^2-ab+b^2=x, b^2-bc+c^2=y, c^2-ca+a^2=z \text{ and then :}$$

$$(c^2-ca+a^2)-(b^2-bc+c^2)=(a-b)(a+b)-c(a-b)$$

$$=(a-b)(a+b-c)\Rightarrow(z-y)^2=(a-b)^2(a+b-c)^2 \text{ and analogs}$$

$$\therefore B\sum_{cyc}(a-b)^2(a+b-c)^2\big((a-b)^2+ab\big)$$

$$=\frac{(a+b)(b+c)(c+a)\big((z-y)^2x+(z-x)^2y+(x-y)^2z\big)}{(a+b)(a^2-ab+b^2)(b+c)(b^2-bc+c^2)(c+a)(c^2-ca+a^2)}$$

$$=\frac{1}{xyz}\cdot\left(\sum_{cyc}\left(xy\left(\sum_{cyc}x-z\right)\right)-6xyz\right)$$

$$=\frac{(\sum_{cyc}x)(\sum_{cyc}xy)-9xyz}{xyz}\therefore AB\sum_{cyc}(a-b)^2(a+b-c)^2\big((a-b)^2+ab\big)$$

$$=\frac{\sum_{cyc}a^3}{2\sum_{cyc}a^2-\sum_{cyc}ab}\cdot\left(\frac{(\sum_{cyc}x)(\sum_{cyc}xy)}{xyz}-9\right)$$

$$=\frac{\sum_{cyc}a^3}{\sum_{cyc}x}\cdot\frac{(\sum_{cyc}x)(\sum_{cyc}xy)}{xyz}-\frac{9\sum_{cyc}a^3}{2\sum_{cyc}a^2-\sum_{cyc}ab}$$

$$\therefore AB\sum_{cyc}(a-b)^2(a+b-c)^2\big((a-b)^2+ab\big)$$

$$=\frac{(\sum_{cyc}a^3)(\sum_{cyc}xy)}{xyz}-\frac{9\sum_{cyc}a^3}{2\sum_{cyc}a^2-\sum_{cyc}ab}\rightarrow(1)$$

$$\text{Again,}\sum_{cyc}\frac{a^3}{b^2-bc+c^2}=\frac{a^3zx+b^3xy+c^3yz}{xyz}\rightarrow(2)\therefore(1),(2)$$

$$\Rightarrow\sum_{cyc}\frac{a^3}{b^2-bc+c^2}\geq a+b+c+AB\sum_{cyc}(a-b)^2(a+b-c)^2\big((a-b)^2+ab\big)$$

$$\Leftrightarrow \frac{(\sum_{cyc} a^3)(\sum_{cyc} xy) - (a^3zx + b^3xy + c^3yz)}{xyz} + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0$$

$$\Leftrightarrow \frac{a^3xy + a^3yz + b^3yz + b^3zx + c^3xy + c^3zx}{xyz} + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0$$

$$\Leftrightarrow \frac{xy(c+a)z + yz(a+b)x + zx(b+c)y}{xyz} + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0$$

$$\Leftrightarrow 2\sum_{cyc} a + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0 \Leftrightarrow \frac{3\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \ge \sum_{cyc} a$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3abc \ge \sum_{cyc} a^2b + \sum_{cyc} ab^2 \rightarrow \text{true via Schur}$$

$$\therefore \sum_{cyc} \frac{a^3}{b^2 - bc + c^2} \ge a + b + c + AB\sum_{cyc} (a-b)^2(a+b-c)^2\left((a-b)^2 + ab\right)$$

$\forall\, a, b, c \ge 0$ with $ab + bc + ca \ne 0$, $'' = ''$ iff $(a = b = c)$ or $(a = 0, b = c \ne 0)$ or $(b = 0, c = a \ne 0)$ or $(c = 0, a = b \ne 0)$ (QED)

1391. If $a, b, c, d > 0$ such that $a + b + c + d = 1$ then

$$\sum_{cyc} \frac{a}{a^3 + b^4 + c^4 + d^4} \le \frac{1}{7abcd}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have $1 = a + b + c + d \ge 4\sqrt[4]{abcd}$, then

$$abcd \le \frac{1}{4^4}, \text{and}$$

$$\frac{a^2bcd}{a^3 + b^4 + c^4 + d^4} = \frac{a^2bcd}{4.\frac{a^3}{4} + b^4 + c^4 + d^4} \le \frac{a^2bcd}{7\sqrt[7]{\left(\frac{a^3}{4}\right)^4 . b^4 . c^4 . d^4}} = \frac{\sqrt[7]{4^4a^2b^3c^3d^3}}{7}$$

$$\le \frac{\sqrt[7]{4^4\left(\frac{1}{4^4}\right)^2 bcd}}{7} \le \frac{4.\frac{1}{4} + b + c + d}{7.7} = \frac{b + c + d + 1}{49}.$$

Therefore

$$\sum_{cyc} \frac{a}{a^3 + b^4 + c^4 + d^4} \le \sum_{cyc} \frac{b + c + d + 1}{49abcd} = \frac{3(a + b + c + d) + 4}{49abcd} = \frac{1}{7abcd}.$$

1392. If $a,b>0$ then:

$$\frac{2ab}{a+b}+\sqrt[6]{4(a^3+b^3)}.\sqrt{a+b}\le\frac{3(a+b)}{2}$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By $AM-GM$ inequality, we have

$$\sqrt[3]{4(a^3+b^3)(a+b)^3}\le\frac{(a+b)^2+(a+b)^2+4(a^2-ab+b^2)}{3}=2(a^2+b^2).$$

So it suffices to prove that

$$\frac{2ab}{a+b}+\sqrt{2(a^2+b^2)}\le\frac{3(a+b)}{2},$$

which is equivalent to

$$\sqrt{2(a^2+b^2)}-(a+b)\le\frac{a+b}{2}-\frac{2ab}{a+b}\text{ or }\frac{(a-b)^2}{\sqrt{2(a^2+b^2)}+(a+b)}\le\frac{(a-b)^2}{2(a+b)},$$

which is true by $AM-QM$ inequality : $\sqrt{2(a^2+b^2)}\ge a+b$.

Equality holds iff $a=b$.

1393. If $a,b,c>0$ and $n\in\mathbb{N}^*$, then :

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{n^2}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\left(1+\frac{1}{n}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{n^2}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\left(1+\frac{1}{n}\right)^2$$

$$\Leftrightarrow\sum_{cyc}\frac{a}{a+b}+\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1+1\right)\ge1+\frac{2}{n}+\frac{1}{n^2}$$

$$\Leftrightarrow\left(\sum_{cyc}\frac{a}{a+b}-1\right)+\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1\right)\overset{(*)}{\ge}\frac{2}{n}$$

Now, $\left(\sum_{cyc}\frac{a}{a+b}-1\right)+\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1\right)\overset{A-G}{\ge}\frac{2}{n}.\sqrt{\left(\sum_{cyc}\frac{a}{a+b}-1\right)\left(\sum_{cyc}\frac{b}{a}-1\right)}$

$$=\frac{2}{n}\cdot\sqrt{\left(\sum_{cyc}\frac{1}{1+x}-1\right)\left(\sum_{cyc}x-1\right)}\left(\text{where }\frac{b}{a}=x,\frac{c}{b}=y,\frac{a}{c}=z\right)\overset{?}{\geq}\frac{2}{n}$$

$$\Leftrightarrow\left(\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}-1\right)(x+y+z-1)\overset{?}{\geq}1$$

$$\Leftrightarrow\sum_{cyc}x^2+\sum_{cyc}xy\overset{?}{\geq}xyz\sum_{cyc}x+3\Leftrightarrow\sum_{cyc}x^2+\sum_{cyc}xy\underset{(**)}{\overset{?}{\geq}}\sum_{cyc}x+3\ (\because xyz=1)$$

$$\text{Now,}\sum_{cyc}x^2\geq\frac{1}{3}\left(\sum_{cyc}x\right)^2=\frac{1}{3}\left(\sum_{cyc}\frac{b}{a}\right)\left(\sum_{cyc}x\right)\overset{A-G}{\geq}\frac{1}{3}.3.\sum_{cyc}x\Rightarrow\sum_{cyc}x^2\geq\sum_{cyc}x$$

$$\to(1)\text{ and }\sum_{cyc}xy\overset{A-G}{\geq}3.\sqrt[3]{(xyz)^2}=3\ (\because xyz=1)\to(2)\therefore(1)+(2)$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{n^2}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq\left(1+\frac{1}{n}\right)^2$$

$\forall\, a,b,c>0$ and $n\in\mathbb{N}^*$,'' = '' if $a=b=c$ and if

$$\left(\sum_{cyc}\frac{a}{a+b}-1\right)=\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1\right)\Rightarrow\text{if }n=2$$

i.e.,'' = '' iff $(a=b=c$ and $n=2)$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=a^2+b^2+c^2$ and $y:=ab+bc+ca$. By CBS inequality, we have

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\geq\frac{(a+b+c)^2}{\sum_{cyc}a(a+b)}=\frac{x+2y}{x+y}=1+\frac{y}{x+y}.$$

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\geq\frac{(b+c+a)^2}{ba+cb+ac}=\frac{x+2y}{y}=\frac{x+y}{y}+1.$$

Using these inequalities, we have

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{n^2}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq1+\left(\frac{y}{x+y}+\frac{x+y}{n^2y}\right)+\frac{1}{n^2}$$

$$\overset{AM-GM}{\geq}1+\frac{2}{n}+\frac{1}{n^2}=\left(1+\frac{1}{n}\right)^2.$$

Equality holds iff $a=b=c$ and $\frac{y}{x+y}=\frac{x+y}{n^2y}\Leftrightarrow a=b=c$ and $n=2$.

1394. If $a_1,a_2,\dots,a_n>0$, then :

$$\sum_{cyc}\frac{(a_1+a_2)^6}{(a_1a_2)^2}\geq64(a_1^2+a_2^2+\dots+a_n^2)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(a_1+a_2)^6}{(a_1a_2)^2}=\frac{(a_1+a_2)^4(a_1+a_2)^2}{(a_1a_2)^2}=\frac{\left(a_1^2+a_2^2+2a_1a_2\right)^2(a_1+a_2)^2}{a_1^2a_2^2}$$

$$\overset{A-G}{\geq}\frac{4\left(a_1^2+a_2^2\right)(2a_1a_2)(4a_1a_2)}{a_1^2a_2^2}\Rightarrow\frac{(a_1+a_2)^6}{(a_1a_2)^2}\geq 32\left(a_1^2+a_2^2\right)\text{ and analogs}$$

$$\therefore\sum_{cyc}\frac{(a_1+a_2)^6}{(a_1a_2)^2}\geq 64\left(a_1^2+a_2^2+\cdots+a_n^2\right),''=''\text{ iff }a_1=a_2=\cdots=a_n\text{ (QED)}$$

1395. If $a,b,c\geq 0$, then :

$$a^2+b^2+c^2+2abc+1\geq 2(ab+bc+ca)+\left(2\sqrt[3]{abc}+1\right)\left(\sqrt[3]{abc}-1\right)^2$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ $a=b=c=0$ and then : LHS − RHS = 1 − 1 ⇒ LHS = RHS

$\boxed{\text{Case 2}}$ Exactly 2 variables = 0 and WLOG we may assume $b=c=0$ $(a>0)$ and then : LHS − RHS $=a^2+1-1>0\Rightarrow$ LHS > RHS

$\boxed{\text{Case 3}}$ Exactly 1 variable = 0 and WLOG we may assume $a=0$ $(b,c>0)$ and then : LHS − RHS $=b^2+c^2+1-2bc-1=(b-c)^2\geq 0\Rightarrow$ LHS ≥ RHS

$\boxed{\text{Case 4}}$ $a,b,c>0$ and let $\sqrt[3]{a}=x,\sqrt[3]{b}=y,\sqrt[3]{c}=z$ and then :

$$a^2+b^2+c^2+2abc+1\geq 2(ab+bc+ca)+\left(2\sqrt[3]{abc}+1\right)\left(\sqrt[3]{abc}-1\right)^2$$

$$\Leftrightarrow\sum_{cyc}x^6+3x^2y^2z^2\geq 2\sum_{cyc}x^3y^3$$

$$\Leftrightarrow 3x^2y^2z^2+\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^4-\sum_{cyc}x^2y^2\right)+3x^2y^2z^2$$

$$\geq 6x^2y^2z^2+2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2y^2-xyz\sum_{cyc}x\right)$$

$$\Leftrightarrow\boxed{\left(\sum_{cyc}x^2\right)\left(\left(\sum_{cyc}x^2\right)^2-3\sum_{cyc}x^2y^2\right)\overset{(*)}{\geq}2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2y^2-xyz\sum_{cyc}x\right)}$$

Assigning $y+z=X,z+x=Y,x+y=Z\Rightarrow X+Y-Z=2z>0,Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z,Y+Z>X,Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius = s, R, r (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (1) \Rightarrow x = s - X, y = s - Y,$$

$$z = s - Z \text{ and such substitutions} \Rightarrow \sum_{cyc} xy = \sum_{cyc}(s-X)(s-Y)$$

$$\Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (2), \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy$$

$$\overset{\text{via (1) and (2)}}{=} s^2 - (4Rr + r^2) \Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \to (3) \text{ and}$$

$$xyz = (s-X)(s-Y)(s-Z) = r^2 s \to (4) \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x$$

$$\overset{\text{via (1),(2) and (4)}}{=} (4Rr + r^2)^2 - 2r^2 s.s \Rightarrow \sum_{cyc} x^2y^2 = (4Rr + r^2)^2 - 2r^2s^2 \to (5)$$

$$\therefore (*) \Leftrightarrow (s^2 - 8Rr - 2r^2)\left((s^2 - 8Rr - 2r^2)^2 - 3\left((4Rr + r^2)^2 - 2r^2s^2\right)\right)$$

$$\geq (4Rr + r^2)\left((4Rr + r^2)^2\right.$$

$$\left. - 3r^2s^2\right) \boxed{s^6 - 24Rrs^4 + r^2s^2(144R^2 + 48Rr + 3r^2) - 4r^3(4R + r)^3 \overset{(**)}{\geq} 0} \text{ and}$$

$$\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$\text{LHS of } (**) \geq (s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (24Rr - 15r^2)s^4 - r^2s^2(624R^2 - 528Rr + 72r^2)$$

$$+ r^3(3840R^3 - 4032R^2r + 1152Rr^2 - 129r^3) \overset{(***)}{\geq} 0 \text{ and}$$

$$\because (24Rr - 15r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \geq (24Rr - 15r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (24R^2 - 32Rr + 13r^2)s^2 \overset{(****)}{\geq} r(384R^3 - 608R^2r + 308Rr^2 - 41r^3)$$

$$\text{Now, LHS of } (****) \overset{\text{Rouche}}{\geq} (24R^2 - 32Rr + 13r^2)\left(\begin{matrix} 2R^2 + 10Rr - r^2 \\ -2(R-2r).\sqrt{R^2 - 2Rr}\end{matrix}\right)$$

$$\overset{?}{\geq} r(384R^3 - 608R^2r + 308Rr^2 - 41r^3)$$

$$\Leftrightarrow \boxed{(R-2r)(24R^3 - 56R^2r + 33Rr^2 - 7r^3) \underset{(*****)}{\overset{?}{\geq}} (R-2r)(24R^2 - 32Rr + 13r^2).\sqrt{R^2 - 2Rr}}$$

$$\text{Now, } 24R^3 - 56R^2r + 33Rr^2 - 7r^3 = (R-2r)(24R^2 - 8Rr + 17r^2) + 27r^3 \overset{\text{Euler}}{\geq}$$

$$27r^3 > 0 \text{ and } \because R - 2r \overset{\text{Euler}}{\geq} 0 \therefore \text{in order to prove } (*****), \text{it suffices to prove :}$$

$$24R^3 - 56R^2r + 33Rr^2 - 7r^3 > (24R^2 - 32Rr + 13r^2).\sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow (24R^3 - 56R^2r + 33Rr^2 - 7r^3)^2 > (R^2 - 2Rr)(24R^2 - 32Rr + 13r^2)^2$$

$$\Leftrightarrow 96R^3 + 40R^2r - 124Rr^2 + 49r^3 > 0$$

$$\Leftrightarrow \boxed{65R^3 + 40R^2r + 31R(R^2 - 4r^2) + 49r^3 > 0} \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*****)$$

$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ **is true** $\therefore a^2+b^2+c^2+2abc+1 \geq$
$2(ab+bc+ca)+\left(2\sqrt[3]{abc}+1\right)\left(\sqrt[3]{abc}-1\right)^2 \ \forall\, a,b,c>0 \therefore$ *combining all* **cases,**
$$a^2+b^2+c^2+2abc+1 \geq 2(ab+bc+ca)+\left(2\sqrt[3]{abc}+1\right)\left(\sqrt[3]{abc}-1\right)^2$$
$\forall\, a,b,c \geq 0, ''='' $ **iff** $(a=b=c)$ **or** $(a=0, b=c\neq 0)$ **or** $(b=0, c=a\neq 0)$
or $(c=0, a=b\neq 0)$ **(QED)**

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows
$$a^2+b^2+c^2+3\sqrt[3]{(abc)^2} \geq 2(ab+bc+ca).$$
By Schur's inequality on the triple $\left(\sqrt[3]{a^2},\sqrt[3]{b^2},\sqrt[3]{c^2}\right)$**, we have**
$$a^2+b^2+c^2+3\sqrt[3]{(abc)^2}$$
$$\geq \sqrt[3]{a^2b^2}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)+\sqrt[3]{b^2c^2}\left(\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)+\sqrt[3]{c^2a^2}\left(\sqrt[3]{c^2}+\sqrt[3]{a^2}\right).$$
Also, by AM − GM inequality, we have
$$\sqrt[3]{a^2}+\sqrt[3]{b^2} \geq 2\sqrt[3]{ab} \ \text{(and analogs)}.$$
Therefore
$$a^2+b^2+c^2+3\sqrt[3]{(abc)^2} \geq 2(ab+bc+ca).$$
Equality holds iff $(a=b=c)$ **or** $(a=0, b=c)$ **and permutation.**

1396. **If** $a,b,c,\lambda>0$ **then :**
$$(a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2 \geq \frac{ab+bc+ca}{\lambda+1}-3\lambda$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{a}{\lambda+1}, y=\frac{b}{\lambda+1}, z=\frac{c}{\lambda+1} \text{ and then :}$$
$$\boxed{(a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2-\left(\frac{ab+bc+ca}{\lambda+1}-3\lambda\right)}$$
$$=(\lambda+1)^2\sum_{cyc}\left(x-\frac{\lambda}{\lambda+1}\right)^2-(\lambda+1)\sum_{cyc}xy+3\lambda$$
$$\geq \frac{(\lambda+1)^2}{3}\cdot\left(\sum_{cyc}x-\frac{3\lambda}{\lambda+1}\right)^2-\frac{(\lambda+1)}{3}\cdot\left(\sum_{cyc}x\right)^2+3\lambda$$
$$=\frac{1}{3}\left((\lambda+1)^2\left(t^2-\frac{6\lambda t}{\lambda+1}+\frac{9\lambda^2}{(\lambda+1)^2}\right)-(\lambda+1)t^2+9\lambda\right)\left(t=\sum_{cyc}x\right)$$

$$=\frac{1}{3}\left(\left((\lambda+1)^2-(\lambda+1)\right)t^2-6\lambda(\lambda+1)t+9\lambda^2+9\lambda\right)$$

$$=\frac{\lambda^2+\lambda}{3}.\left(t^2-6t+9\right)\boxed{=\frac{\lambda^2+\lambda}{3}.(t-3)^2\geq 0}(\because \lambda>0)$$

$$\therefore (a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2\geq \frac{ab+bc+ca}{\lambda+1}-3\lambda\ \forall\ a,b,c,\lambda>0,$$

$$''='' \text{ iff } x=y=z=1\Rightarrow \text{iff } \frac{a}{\lambda+1}=\frac{b}{\lambda+1}=\frac{c}{\lambda+1}=1$$

$$\Rightarrow \text{iff } a=b=c=\lambda+1\ (\text{QED})$$

1397. If $a,b>0$ then:

$$\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\geq 2+\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $a\geq b$. Denote $x=\frac{a}{b}\geq 1$.

$$\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\geq 2+\sqrt{2},\qquad \frac{a}{b}+\frac{b}{a}+\frac{\frac{a}{b}+1}{\sqrt{\left(\frac{a}{b}\right)^2+1}}\geq 2+\sqrt{2}$$

$$x+\frac{1}{x}+\frac{x+1}{\sqrt{x^2+1}}\geq 2+\sqrt{2}$$

Let be:

$$f(x)=x+\frac{1}{x}+\frac{x+1}{\sqrt{x^2+1}},\qquad f'(x)=1-\frac{1}{x^2}+\frac{\sqrt{x^2+1}-\frac{(x+1)x}{\sqrt{x^2+1}}}{x^2+1}$$

$$f'(x)=\frac{(x-1)(x+1)}{x^2}+\frac{x^2+1-(x+1)x}{(x^2+1)\sqrt{x^2+1}}$$

$$f'(x)=(x-1)\left(\frac{x+1}{x^2}-\frac{1}{(x^2+1)\sqrt{x^2+1}}\right)$$

We will prove that:

$$\frac{x+1}{x^2}-\frac{1}{(x^2+1)\sqrt{x^2+1}}>0,\qquad x>0$$

$$\frac{x+1}{x^2}>\frac{1}{(x^2+1)\sqrt{x^2+1}},\qquad (x+1)(x^2+1)\sqrt{x^2+1}>x^2$$

$$(x+1)^2(x^2+1)^3 > x^4$$

$$(x+1)^2(x^2+1)^3 \overset{AM-GM}{\geq} \left(2\sqrt{x}\right)^2(2x)^3 = 32x^4 > x^4$$

$$f'(x) \geq 0 \Rightarrow f \text{ −increasing},$$

$$\underbrace{minf(x)}_{x\geq 1} = f(1) = 2+\sqrt{2} \Rightarrow f(x) \geq 2+\sqrt{2}$$

Equality holds for $x = 1 \Leftrightarrow a = b$.

1398. If $a, b, c > 1$ then:

$$log_a(bc) + log_b(ca) + log_c(ab) \geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Let be:

$$x = lna, y = lnb, z = lnc$$

$$a, b, c > 1 \Rightarrow x, y, z > 0$$

$$log_a(bc) + log_b(ca) + log_c(ab) = \frac{ln(bc)}{lna} + \frac{ln(ca)}{lnb} + \frac{ln(ab)}{lnc} =$$

$$= \frac{lnb + lnc}{lna} + \frac{lnc + lna}{lnb} + \frac{lna + lnb}{lnc} = \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} =$$

$$= \left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{z}{x} + \frac{x}{z}\right) \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{\frac{x}{y}\cdot\frac{y}{x}} + 2\sqrt{\frac{y}{z}\cdot\frac{z}{y}} + 2\sqrt{\frac{z}{x}\cdot\frac{x}{z}} = 2+2+2 = 6$$

Equality holds for $a = b = c$.

1399. If $a, b, c > 0$, *then* :

$$\frac{81abc}{\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)^3} + 14\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq 24$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius = s, R, r (say)

$$\text{yielding } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3) \therefore \frac{81abc}{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^3} + 14\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

$$\overset{\substack{\text{CBS}\\ \text{and}\\ \text{via (2)}}}{\geq} \frac{81r^2 s}{\left(\sqrt{3\left(\sum_{cyc} ab\right)}\right)^3} + 14\sum_{cyc}\frac{s-x}{x} \overset{\text{via (3)}}{=} \frac{81r^2 s}{3(4Rr+r^2).\sqrt{3(4Rr+r^2)}}$$

$$+14\left(\frac{s\sum_{cyc} xy}{xyz} - 3\right) \geq \frac{81r^2 s}{3(4Rr+r^2).s} + 14\left(\frac{s(s^2+4Rr+r^2)}{4Rrs} - 3\right)$$

$$\left(\begin{matrix} \because s^2 - 12Rr - 3r^2 = s^2 - 16Rr + 5r^2 + 4r(R-2r) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\geq} 0 \\ \Rightarrow 3(4Rr+r^2) \leq s^2 \end{matrix}\right)$$

$$= \frac{27r}{4R+r} + \frac{7(s^2-8Rr+r^2)}{2Rr} \overset{?}{\geq} 24 \Leftrightarrow \frac{7(4R+r)(s^2-8Rr+r^2)+54Rr^2}{2Rr(4R+r)} \overset{?}{\geq} 24$$

$$\Leftrightarrow (28R+7r)s^2 \underset{(*)}{\overset{?}{\geq}} r(416R^2+22Rr-7r^2)$$

$$\text{Now, } (28R+7r)s^2 \overset{\text{Gerretsen}}{\geq} (28R+7r)(16Rr-5r^2) \overset{?}{\geq} r(416R^2+22Rr-7r^2)$$

$$\Leftrightarrow 16R^2 - 25Rr - 14r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(16R+7r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{81abc}{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^3} + 14\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 24$$

$$\forall a,b,c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM and CBS inequality, we have

$$(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^3 \leq 3(ab+bc+ca)(a+b+c).$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)} = \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Using these inequalities, we have

$$\frac{81abc}{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^3} + 14\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

$$\geq \frac{27abc}{(ab+bc+ca)(a+b+c)} + \frac{7(a+b+c)^2}{ab+bc+ca}$$

$$= \frac{27abc+7(a+b+c)^3}{(ab+bc+ca)(a+b+c)} \overset{?}{\geq} 24$$

$$\Leftrightarrow 7(a^3+b^3+c^3)\ge 3ab(a+b)+3bc(b+c)+3ca(c+a)+3abc,$$

which is true by AM − GM inequality :

$$ab(a+b)\le (a^2+b^2-ab)(a+b)=a^3+b^3 \text{ (and analogs) and } 3abc \le a^3+b^3+c^3.$$

So the proof is complete. Equality holds iff $a=b=c$.

1400.

If $0<x,y,z<2$ with $xy+yz+zx=3$ and $\lambda\ge 3$ then :

$$\sqrt{\frac{yz}{x^3-x+\lambda}}+\sqrt{\frac{zx}{y^3-y+\lambda}}+\sqrt{\frac{xy}{z^3-z+\lambda}}\le\frac{3}{\sqrt{\lambda}}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove the lemma that for all $x\in(0,2)$ and $\lambda\ge 3$, we have

$$\frac{1}{x^3-x+\lambda}\le\frac{\lambda+2-2x}{\lambda^2}.$$

The inequality is equivalent to

$$(\lambda+2-2x)(x^3-x+\lambda)-\lambda^2\ge 0 \text{ or } -2x^4+(\lambda+2)x^3+2x^2-(3\lambda+2)x+2\lambda\ge 0$$

$$\text{or } (x-1)^2[(\lambda-3)(2+x)+(2-x)(3+2x)]\ge 0,$$

which is true and the proof of the lemma is complete.

Now, by using the CBS inequality and the lemma above, we have

$$\sum_{cyc}\sqrt{\frac{yz}{x^3-x+\lambda}}\le\sqrt{\sum_{cyc}yz.\sum_{cyc}\frac{1}{x^3-x+\lambda}}\le\sqrt{3\sum_{cyc}\frac{\lambda+2-2x}{\lambda^2}}$$

$$=\sqrt{3.\frac{3(\lambda+2)-2(x+y+z)}{\lambda^2}}$$

$$\le\sqrt{3.\frac{3\lambda+6-2\sqrt{3(xy+yz+zx)}}{\lambda^2}}=\frac{3}{\sqrt{\lambda}},$$

as desired. Equality holds iff $x=y=z=1$.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru