

# Some Algebraic Inequality Lemmas and Their Application on Problems

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## 1 Introduction

**Lemma 1.1**  $a, b, c, k \in \mathbf{R}^+$  such that  $a + b + c = x$  and  $k \geq 1$ . Then prove that

$$\frac{a^k + k - 1}{b + k - 1} + \frac{b^k + k - 1}{c + k - 1} + \frac{c^k + k - 1}{a + k - 1} \geq \frac{3xk}{x + 3k - 3}$$

*Proof.* By effective AM-GM and Bergstrom Inequality

$$\begin{aligned} \sum_{cyc} \frac{a^k + k - 1}{b + k - 1} &= \sum_{cyc} \frac{\overbrace{a^k + 1 + 1 + \dots + 1}^k}{b + k - 1} \stackrel{AM-GM}{\geq} \sum_{cyc} \frac{ka}{b + k - 1} \\ &= k \left( \frac{a}{b + k - 1} \right) \stackrel{Bergstorm}{\geq} \frac{k(a + b + c)^2}{ab + ac + bc + (k - 1)(a + b + c)} \\ &= k \left( \frac{a + b + c}{\frac{a+b+c}{3} + k - 1} \right) = \frac{3xk}{x + 3k - 3} \end{aligned}$$

### Application on Problem- Macedonia 2010 1.2

$a, b, c \in \mathbf{R}^+$  such that  $a + b + c = 3$ . Then prove that

$$\frac{a^3 + 2}{b + 2} + \frac{b^3 + 2}{c + 2} + \frac{c^3 + 2}{a + 2} \geq 3$$

*Proof.* By **Lemma 1.1**, placing  $x = 3$  and  $k = 3$ , we get the following :

$$\frac{a^3 + 2}{b + 2} + \frac{b^3 + 2}{c + 2} + \frac{c^3 + 2}{a + 2} \geq \frac{3xk}{x + 3k - 3} = 3$$

**Lemma 2.1.**

$x, y, z, a, b, c \in \mathbf{R}^+$  such that  $x + y + z = k$ . Then prove that

$$\frac{1}{\sqrt{x(ay + bz)}} + \frac{1}{\sqrt{y(az + bx)}} + \frac{1}{\sqrt{z(ax + by)}} \geq \frac{9}{k\sqrt{a + b}}$$

*Proof.* By Bergstorm Inequality and Jensen Inequality

$$\begin{aligned} \sum_{cyc} \frac{1}{\sqrt{x(ay + bz)}} &\stackrel{Bergstrom}{\geq} \frac{9}{\sqrt{x(ay + bz)} + \sqrt{y(az + bx)} + \sqrt{z(ax + by)}} \\ &\stackrel{Jensen}{\geq} \frac{9}{3\sqrt{\frac{axy + ayz + axz + bxy + byz + bxz}{3}}} = \frac{9}{3\sqrt{\frac{(a+b)(xy + yz + zx)}{3}}} \\ &\geq \frac{9}{3\sqrt{\frac{(a+b)(\frac{(x+y+z)^2}{3})}{3}}} = 3\sqrt{\frac{9}{k^2(a+b)}} = \frac{9}{k\sqrt{a+b}} \end{aligned}$$

**Lemma 2.2.**

$x, y, z \in \mathbf{R}^+$  such that  $x + y + z = 3$ . Then prove that

$$\frac{1}{\sqrt{x(2y + 3z)}} + \frac{1}{\sqrt{y(2z + 3x)}} + \frac{1}{\sqrt{z(2x + 3y)}} \geq \frac{3}{k\sqrt{5}}$$

*Proof.*

By **Lemma 2.1.**, placing  $x + y + z = 3$  and  $a + b = 5$ , we get the following:

$$\frac{1}{\sqrt{x(2y+3z)}} + \frac{1}{\sqrt{y(2z+3x)}} + \frac{1}{\sqrt{z(2x+3y)}} \geq \frac{3}{k\sqrt{5}} \geq \frac{9}{k\sqrt{a+b}} = \frac{3}{\sqrt{5}}$$

**Lemma 3.1.**  $y_1, y_2, \dots, y_n, x, p, n \in \mathbf{R}^+$  positive reals and  $n > p \geq 2$  integers,  $a_1, a_2, \dots, a_n \geq 1$  integers,  $m \geq 1$ . Then prove that

$$\sum_{y_i, i=1}^{i=p} \binom{n-1}{p-1} \sqrt{\frac{xy_1^{\frac{4m(n-1)!}{(n-p)!}} + k^2x}{\prod_{a_1+a_2+\dots+a_p=n} (a_1y_1 + a_2y_2 + \dots + a_p y_p)}} \geq \binom{n-1}{p-1} \sqrt{2xk} \cdot \binom{n+1}{p-1} \frac{2 \cdot \left(\frac{y_1+y_2+\dots+y_n}{p}\right)^{2m(p-1)-1}}{n(n+1)}$$

Determine when does the equality occurs ?

*Proof* By effective AM-GM and Bergstrom

Let us show

$$xy_1^t + k^2x \geq 2xky_1^{\frac{y}{2}}$$

.

$$xy_1^t + k^2x = \overbrace{\frac{y_1^t}{k} + \frac{y_1^t}{k} + \dots + \frac{y_1^t}{k}}^{xk} + \overbrace{k + k + \dots + k}^{xk} \geq 2xky^{\frac{txk}{2xk}} = 2xky^{\frac{t}{2}}$$

Then for the denominator

$$xy_1^{\frac{4m(n-1)!}{(n-p)!}} + k^2x \geq 2xky_1^{\frac{2m(n-1)!}{(n-p)!}}$$

$$\begin{aligned} & \sum_{y_i, i=1}^{i=p} \binom{n-1}{p-1} \sqrt{\frac{xy_1^{\frac{4m(n-1)!}{(n-p)!}} + k^2x}{\prod_{a_1+a_2+\dots+a_p=n} (a_1y_1 + a_2y_2 + \dots + a_p y_p)}} \\ & \geq \sum_{y_i, i=1}^{i=p} \binom{n-1}{p-1} \sqrt{\frac{2xky_1^{\frac{2m(n-1)!}{(n-p)!}}}{\prod_{a_1+a_2+\dots+a_p=n} (a_1y_1 + a_2y_2 + \dots + a_p y_p)}} \end{aligned}$$

$$\begin{aligned}
&= \frac{\binom{n-1}{p-1} \sqrt{2xk} \left( y_1^{2m(p-1)!} + y_2^{2m(p-1)!} + \dots + y_p^{2m(p-1)!} \right)}{\binom{n-1}{p-1} \sqrt{\prod_{a_1+a_2+\dots+a_p=n} (a_1 y_1 + a_2 y_2 + \dots + a_p y_p)}} \\
&\geq \frac{\binom{n-1}{p-1} \sqrt{2xk} \cdot 2 \left( y_1^{2m(p-1)!} + y_2^{2m(p-1)!} + \dots + y_p^{2m(p-1)!} \right)}{\left[ \frac{(n-p+1)(n-p+2)(y_1+y_2+\dots+y_p)}{\binom{n-1}{p-1}} \right]} \\
&\geq \frac{\binom{n-1}{p-1} \sqrt{2xk} \cdot 2 \left( \frac{(y_1+y_2+\dots+y_p)^{2m(p-1)!}}{p^{2m(p-1)!} \cdot p} \right)}{\left[ \frac{(n-p+1)(n-p+2)(y_1+y_2+\dots+y_p)}{\binom{n-1}{p-1}} \right]} \\
&= \binom{n-1}{p-1} \sqrt{2xk} \cdot 2 \left[ \frac{\binom{n-1}{p-1}}{(n-p+1)(n-p+2)} \frac{(y_1+y_2+\dots+y_p)^{2m(p-1)!-1}}{p^{2m(p-1)!-1}} \right] \\
&= \binom{n-1}{p-1} \sqrt{2xk} \cdot \frac{(n-1)!}{(n-p)!(n-p+1)(n-p+2)(p-1)!} \cdot 2 \left( \frac{y_1+y_2+\dots+y_p}{p} \right)^{2m(p-1)!-1} \\
&= \binom{n-1}{p-1} \sqrt{2xk} \cdot \frac{(n-1)!}{(n-p+2)!(p-1)!} \cdot 2 \left( \frac{y_1+y_2+\dots+y_p}{p} \right)^{2m(p-1)!-1} \\
&= \binom{n-1}{p-1} \sqrt{2xk} \cdot \frac{(n+1)!}{(n-p+2)!(p-1)!} \cdot \frac{(n-1)!}{(n+1)!} \cdot 2 \left( \frac{y_1+y_2+\dots+y_p}{p} \right)^{2m(p-1)!-1}
\end{aligned}$$

$$\begin{aligned}
&= \binom{n-1}{p-1} \sqrt[2xk]{2xk} \cdot \binom{n+1}{p-1} \cdot \frac{1}{n(n+1)} \cdot 2 \left( \frac{y_1 + y_2 + \dots + y_p}{p} \right)^{2m(p-1)!-1} \\
&= \binom{n-1}{p-1} \sqrt[2xk]{2xk} \cdot \binom{n+1}{p-1} \cdot 2 \frac{\left( \frac{y_1 + y_2 + \dots + y_n}{p} \right)^{2m(p-1)!-1}}{n(n+1)}
\end{aligned}$$

*Important Places* We know the  $\pi$  includes how many terms inside of it by binomial coefficients  $\Rightarrow \binom{n-1}{p-1}$ . In the problem,  $a_1, a_2, \dots, a_p \geq 1$  integers which allows us to use ball-pocket binomial formula.) After using GM-AM on the denominator,  $\min_{a_1} = 1$ ,  $\max_{a_1} = n - p + 1$  because of other  $p - 1$  has at least one.

Equality occurs if and if  $y_1^{\frac{4m(n-1)!}{(n-p)!}} = y_2^{\frac{4m(n-1)!}{(n-p)!}} = \dots = y_p^{\frac{4m(n-1)!}{(n-p)!}}$  and  $a_1 = a_2 = \dots = a_p = y_1 = \dots = y_p$

**Lemma 3.2**

$a, b, c, x, n \in \mathbf{R}^+$  positive reals,  $i, j, s \geq 1$  integers,  $n \geq 4$  and  $k \geq 2$ . Then prove that

$$\sum_{k=a}^c \binom{n-1}{2} \sqrt[2xk]{\frac{xa^{4m(n-1)!} + 4x}{\prod_{i+j+s=n} (ia + jb + sc)}} \geq \binom{n-1}{2} \sqrt[4x]{4x} \cdot \binom{n+1}{2} \cdot 2 \frac{\left( \frac{a+b+c}{3} \right)^{4m-1}}{n(n+1)}$$

*Proof.*

By **Lemma 3.1.**, placing  $k = 2$ ,  $p = 3$ , we get the following:

$$\begin{aligned}
\sum_{k=a}^c \binom{n-1}{2} \sqrt[2xk]{\frac{xa^{4m(n-1)!} + 4x}{\prod_{i+j+s=n} (ia + jb + sc)}} &\geq \binom{n-1}{p-1} \sqrt[2xk]{2xk} \cdot \binom{n+1}{p-1} \cdot 2 \frac{\left( \frac{y_1 + y_2 + \dots + y_n}{p} \right)^{2m(p-1)!-1}}{n(n+1)} \\
&= \binom{n-1}{2} \sqrt[4x]{4x} \cdot \binom{n+1}{2} \cdot 2 \frac{\left( \frac{a+b+c}{3} \right)^{4m-1}}{n(n+1)}
\end{aligned}$$

**Lemma 3.3**

$a, b, c, x, n \in \mathbf{R}^+$  such that  $n \geq 3$ . Then prove that

$$\sum_{q=a}^c \frac{{}^{(n-1)!} \sqrt[{}]{\frac{xa^{8k(n-1)! + 9x}}{\prod_{i+j=n} (ia + jb)}}}{{}^{(n-1)!} \sqrt[{}]{6x} \cdot (n-2)! \frac{(\frac{a+b+c}{3})^{4k-1}}{6n}} \geq$$

*Proof.*

By  $xa^t + k^2x \geq 2xka^{\frac{t}{2}}$

$$\begin{aligned} & \sum_{cyc} {}^{n-1} \sqrt[{}]{\frac{xa^{8m(n-1)} + 9x}{\prod_{i+j=n} (ia + jb)}} \stackrel{AM-GM}{\geq} \sum_{cyc} {}^{n-1} \sqrt[{}]{6x \cdot \frac{a^{4m(n-1)}}{\prod_{i+j=n} (ia + jb)}} \\ & = {}^{n-1} \sqrt[{}]{6x} \left( \sum_{cyc} {}^{n-1} \sqrt[{}]{\frac{a^{4m(n-1)}}{\prod_{i+j=n} (ia + jb)}} \right) \\ & = {}^{n-1} \sqrt[{}]{6x} \left( \sum_{cyc} \frac{a^{4m}}{{}^{n-1} \sqrt[{}]{\prod_{i+j=n} (ia + jb)}} \right) \stackrel{GM-AM}{\geq} {}^{n-1} \sqrt[{}]{6x} \left( \sum_{cyc} \frac{a^{4m}}{\frac{n(n-1)(a+b)}{(n-1)}} \right) \\ & = {}^{n-1} \sqrt[{}]{6x} \left( \sum_{cyc} \frac{a^{4m}}{n(a+b)} \right) \geq {}^{n-1} \sqrt[{}]{6x} \left( \frac{(a+b+c)^{4m}}{2n(a+b+c) \cdot 3^{4m-2}} \right) \\ & = {}^{n-1} \sqrt[{}]{6x} \left( \frac{(\frac{a+b+c}{3})^{4m-1}}{6n} \right) \end{aligned}$$

**Lemma 4.1.**

$a, b, c, x, t, y \in \mathbf{R}^+$ ,  $a+b+c = k$  and  $x \geq 2$ . Then prove that following inequality holds

$$\frac{t - y\sqrt{a}}{\sqrt{c + xa}} + \frac{t - y\sqrt{b}}{\sqrt{a + xb}} + \frac{t - y\sqrt{c}}{\sqrt{b + xc}} \geq \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{k}{3}} - y)$$

$$\sum_{cyc} \frac{t - \sqrt{a}}{\sqrt{c + xa}} = \sum \frac{t}{\sqrt{c + xa}} - \sum_{cyc} \sqrt{\frac{a}{c + xa}}$$

$$\begin{aligned} \sum_{cyc} \frac{t}{\sqrt{c + xa}} &\stackrel{\text{Bergstorm}}{\geq} t \left( \frac{9}{\sqrt{c + xa} + \sqrt{a + xb} + \sqrt{b + xc}} \right) \stackrel{\text{Jensen}}{\geq} t \left( \frac{9}{3\sqrt{\frac{(x+1)(a+b+c)}{3}}} \right) \\ &= \frac{3t}{\sqrt{(x+1)\left(\frac{k}{3}\right)}} \end{aligned}$$

$$\sum_{cyc} \frac{y\sqrt{a}}{\sqrt{c + xa}} \stackrel{\text{Jensen}}{\leq} 3y\sqrt{\frac{\frac{a}{c+xa} + \frac{b}{a+xb} + \frac{c}{b+xc}}{3}}$$

Let  $d$  satysfie  $\frac{a}{c+xa} + \frac{b}{a+xb} + \frac{c}{b+xc} \leq \frac{1}{d}$ .

$$d[a(a+xb)(b+xc) + b(b+xc)(c+xa) + c(c+xa)(a+xb)] \leq (c+xa)(a+xb)(b+xc)$$

After expanding :

$$d(a(ab + axc + xb^2 + x^2bc) + b(bc + xab + xc^2 + x^2ac) + c(ac + xcb + xa^2 + x^2ab))$$

$$= d[(a^2b + b^2c + c^2a) + 2x(a^2c + b^2a + c^2b) + 3x^2abc]$$

$$= d(a^2b + b^2c + c^2a) + 2dx(a^2c + b^2a + c^2b) + 3dx^2(abc)$$

$$(c + xa)(a + xb)(b + xc) = (c + xa)(ab + xac + xb^2 + x^2bc)$$

$$= abc + xac^2 + xcb^2 + xba^2 + x^2bc^2 + x^2ca^2 + x^2ab^2 + x^3abc$$

$$= x(a^2b + b^2c + c^2a) + x^2(a^2c + b^2a + c^2b) + (x^3 + 1)(abc)$$

That implies

$$d(a^2b + b^2c + c^2a) + 2dx(a^2c + b^2a + c^2b) + 3dx^2(abc) \leq x(a^2b + b^2c + c^2a) + x^2(a^2c + b^2a + c^2b) + (x^3 + 1)(abc)$$

$$0 \leq (x - d)(a^2b + b^2c + c^2a) + (x^2 - 2dx)(a^2c + b^2a + c^2b) + (x^3 - 3dx^2 + 1)(abc)$$

$$(x - d)\left(\sum_{cyc} a^2b\right) + (x^2 - 2dx)\left(\sum_{cyc} a^2c\right) + (x^3 - 3dx^2 + 1)(abc)$$

$\overbrace{\geq}^{AM-GM}$

$$(x - d)(3abc) + (x^2 - 2xd)(3abc) + (x^3 - 3x^2d + 1)abc = abc(3x - 3d + 3x^2 - 6xd + x^3 - 3x^2d + 1)$$

$$= abc(x + 1)^2(x + 1 - 3d) \geq 0$$



Because of  $a, b, c, x \in \mathbf{R}^+$ ,  $x + 1 - 3d$  should be greater than 0.

$$d \leq \frac{x+1}{3}$$

We have found  $d$  in a  $x$  way. Placing this in inequality holds. If we place this in inequality:

$$3y\sqrt{\frac{\frac{a}{c+xa} + \frac{b}{a+xb} + \frac{c}{b+xc}}{3}} \geq 3y\sqrt{\frac{1}{3}} \geq 3\sqrt{\frac{3}{x+1}} = y\frac{3}{\sqrt{x+1}}$$

Unit both inequalities:

$$\sum_{cyc} \frac{t - y\sqrt{a}}{\sqrt{c+xa}} \geq \frac{3t}{\sqrt{\frac{k(x+1)}{3}}} - y\frac{3}{\sqrt{x+1}} = \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - y)$$

**Lemma 4.2.**

$a, b, c, x, t \in \mathbf{R}^+$  and  $a + b + c = k$ . Additionally  $x \geq 2$ . Then prove that

$$\frac{t - \sqrt{a}}{\sqrt{c+xa}} + \frac{t - \sqrt{b}}{\sqrt{a+xb}} + \frac{t - \sqrt{c}}{\sqrt{b+xc}} \geq \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - 1)$$

*Proof.*

If we place  $y = 1$  in **Lemma 4.1.**, we get the following:

$$\frac{t - \sqrt{a}}{\sqrt{c+xa}} + \frac{t - \sqrt{b}}{\sqrt{a+xb}} + \frac{t - \sqrt{c}}{\sqrt{b+xc}} \geq \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - 1) = \frac{3}{\sqrt{x+1}}(t\sqrt{\frac{3}{k}} - 1)$$

**Lemma 4.3.**

$a, b, c \in \mathbf{R}^+$  and  $a + b + c = 3$ . Then prove that

$$\frac{2 - \sqrt{a}}{\sqrt{c + 3a}} + \frac{2 - \sqrt{b}}{\sqrt{a + 3b}} + \frac{2 - \sqrt{c}}{\sqrt{b + 3c}} \geq \frac{3}{2}$$

*Proof.*

Placing  $y = 1$ ,  $t = 2$ ,  $k = 3$  and  $x = 3$ , we get the following:

$$\frac{2 - \sqrt{a}}{\sqrt{c + 3a}} + \frac{2 - \sqrt{b}}{\sqrt{a + 3b}} + \frac{2 - \sqrt{c}}{\sqrt{b + 3c}} \geq \frac{3}{\sqrt{x+1}} (t\sqrt{\frac{3}{k}} - 1) = \frac{3}{2}$$

**Lemma 5.1.**

$a, b, c, x, p, s, k \in \mathbf{R}^+$ ,  $k \geq \frac{1}{2}$  and  $a + b + c = t$ . Then prove that

$$\frac{a^{2k-1} + p}{xb + s} + \frac{b^{2k-1} + p}{xc + s} + \frac{c^{2k-1} + p}{xa + s} \geq \frac{9p + \frac{t^{2k-1}}{3^{2k-3}}}{xt + 3s}$$

*First Proof* (Author).

$$\begin{aligned} & \frac{a^{2k-1} + p}{xb + s} + \frac{b^{2k-1} + p}{xc + s} + \frac{c^{2k-1} + p}{xa + s} = \sum_{cyc} \frac{a^{2k-1}}{xb + s} + p \left( \sum_{cyc} \frac{1}{xb + s} \right) \\ &= \sum_{cyc} \frac{a^{2k}}{xab + as} + p \left( \sum_{cyc} \frac{1}{xb + s} \right) \geq \frac{(a + b + c)^{2k}}{3^{2k-2} \cdot (x(ab + bc + ca) + s(a + b + c))} + p \left( \frac{9}{x(a + b + c) + 3s} \right) \\ &\geq \frac{(a + b + c)^{2k}}{3^{2k-2} \cdot \left( x \left( \frac{(a+b+c)^2}{3} \right) + s(a + b + c) \right)} + p \left( \frac{9}{x(a + b + c) + 3s} \right) \geq \frac{9p + \frac{(a+b+c)^{2k-1}}{3^{2k-3}}}{x(a + b + c) + 3s} = \frac{9p + \frac{t^{2k-1}}{3^{2k-3}}}{xt + 3s} \\ &\frac{(a + b + c)^{2k}}{3^{2k-2} \cdot \left( x \left( \frac{(a+b+c)^2}{3} \right) + s(a + b + c) \right)} \geq \frac{\frac{(a+b+c)^{2k-1}}{3^{2k-3}}}{x(a + b + c) + 3s} \end{aligned}$$

$$\frac{a+b+c}{3(a+b+c)\left(\frac{x(a+b+c)}{3}+s\right)} \geq \frac{1}{x(a+b+c)+3s}$$

$$\frac{1}{3\left(x\left(\frac{a+b+c}{3}\right)+s\right)} \geq \frac{1}{x(a+b+c)+3s}$$

$$\frac{1}{x(a+b+c)+3s} \geq \frac{1}{x(a+b+c)+3s}$$

*Second Proof* (Davros)

$$\sum_{cyc} \frac{p}{xa+s} \geq \frac{9p}{xt+3s}$$

We need to prove:

$$\sum_{cyc} \frac{a^{2k-1}}{xb+s} \geq \frac{t^{2k-1}}{3^{2k-3}(xt+3s)}$$

$$\sum_{cyc} \frac{a^{2k-1}}{xb+s} \stackrel{\text{Rearrangement}}{\geq} \sum_{cyc} \frac{a^{2k-1}}{xa+s}$$

$$\text{Let } f(a) = \frac{a^{2k-1}}{xa+s}.$$

$$f''(a) = \frac{2a^{2k-1}(k(2k-1)x^2a^2 + (2k-1)(2k+1)sa + (2k^2+k)s^2)}{(xa+s)^3} \geq 0$$

That means function is strictly convex on  $k \in [\frac{1}{2}, \infty]$

$$\sum_{cyc} \frac{a^{2k-1}}{xa+s} \geq \frac{t^{2k-1}}{3^{2k-3}(xt+3s)}$$

**Lemma 5.2.**

$a, b, c, x, p, s \in \mathbf{R}^+$  and  $a + b + c = t$ . Then prove that

$$\frac{a^7+2}{xb+s} + \frac{b^7+2}{xc+s} + \frac{c^7+2}{xa+s} \geq \frac{\frac{t^7}{3^5} + 18}{xt+3s}$$

*Proof.*

Placing  $k = 4, p = 2$  in **Lemma 5.1.**, we get the following:

$$\frac{a^7+2}{xb+s} + \frac{b^7+2}{xc+s} + \frac{c^7+2}{xa+s} \geq \frac{9p + \frac{t^{2k-1}}{3^{2k-3}}}{xt+3s} = \frac{\frac{t^7}{3^5} + 18}{xt+3s}$$

**Lemma 5.3.**(Marius Stanean)

$a, b, c \in \mathbf{R}^+$  and  $a + b + c = 3$ . Then prove that

$$\frac{a^3+5}{b+2} + \frac{b^3+5}{c+2} + \frac{c^3+5}{a+2} \geq 6$$

*Proof.*

If we place  $k = 2, x = 1, t = 3, s = 2$  and  $p = 5$ , we get the following:

$$\frac{a^3+5}{b+2} + \frac{b^3+5}{c+2} + \frac{c^3+5}{a+2} \geq \frac{a^{2k-1}+p}{xb+s} + \frac{b^{2k-1}+p}{xc+s} + \frac{c^{2k-1}+p}{xa+s} \geq \frac{9p + \frac{t^{2k-1}}{3^{2k-3}}}{xt+3s} = 6$$