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USAMO INEQUALITIES-GENERALIZATIONS

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1 Introduction

Generalization 1.
Let a, b, c, d, n, k be nonnegative reals, $n > 0$ and $a+b+c+d = n^k$. Then prove that

$$\frac{a}{\dfrac{kb^{k+1}}{n}+n^k}+\frac{b}{\dfrac{kc^{k+1}}{n}+n^k}+\frac{c}{\dfrac{kd^{k+1}}{n}+n^k}+\frac{d}{\dfrac{ka^{k+1}}{n}+n^k}\geq\frac{k(4-n^{k-1})+4}{4k+4}$$

Solution.

$$\frac{a}{\dfrac{kb^{k+1}}{n}+n^k}+\frac{b}{\dfrac{kc^{k+1}}{n}+n^k}+\frac{c}{\dfrac{kd^{k+1}}{n}+n^k}+\frac{d}{\dfrac{ka^{k+1}}{n}+n^k}\geq p$$

$$\frac{a}{\dfrac{kb^{k+1}}{n}+n^k}-\frac{a}{n^k}+\frac{b}{\dfrac{kc^{k+1}}{n}+n^k}-\frac{b}{n^k}+\frac{c}{\dfrac{kd^{k+1}}{n}+n^k}-\frac{c}{n^k}+\frac{d}{\dfrac{ka^{k+1}}{n}+n^k}-\frac{d}{n^k}\geq p-\frac{a+b+c+d}{n^k}=p-1$$

$$=a\left(\frac{1}{\dfrac{kb^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)+b\left(\frac{1}{\dfrac{kc^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)+c\left(\frac{1}{\dfrac{kd^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)+d\left(\frac{1}{\dfrac{ka^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)$$

$$=-\left(\sum_{cyc}\frac{akb^{k+1}}{n^{k+1}\left(n^k+\dfrac{kb^{k+1}}{n}\right)}\right)\geq p-1$$

$$\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}\left(n^k+\frac{kb^{k+1}}{n}\right)}\right)\le 1-p$$

$$\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}\left(n^k+\frac{kb^{k+1}}{n}\right)}\right)=\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}\left(n^k+\underbrace{\frac{b^{k+1}}{n}+\frac{b^{k+1}}{n}+\cdots+\frac{b^{k+1}}{n}}_{k}\right)}\right)$$

$$\overbrace{\le}^{AM-GM}\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}(k+1)b^k}\right)$$

$$=\sum_{cyc}\frac{akb}{n^{k+1}(k+1)}=\frac{k}{n^{k+1}(k+1)}(ab+bc+ca+ad))=\frac{k}{n^{k+1}(k+1)}(a+c)(b+d)$$

$$\overbrace{\le}^{AM-GM}\frac{k\left(\frac{a+b+c+d}{2}\right)^2}{n^{k+1}(k+1)}$$

$$=\frac{kn^{2k}}{4n^{k+1}(k+1)}=\frac{kn^{k-1}}{4(k+1)}\le 1-p$$

$$\to p\le\frac{4k+4-kn^{k-1}}{4k+4}=\frac{k(4-n^{k-1})+4}{4k+4}$$

Proof completed.

Application on USAMO 2017 6

Let a, b, c, d be nonnegative reals such that $a+b+c+d=4$. Find the minimum value of

$$\frac{a}{b^3+4}+\frac{b}{c^3+4}+\frac{c}{d^3+4}+\frac{d}{a^3+4}$$

Solution.
This problem is a special case of *Generalization 1.*. The problem denotes case $n = k = 2$. By that

$$\sum_{cyc} \frac{a}{\frac{kb^{k+1}}{n} + n^k} \geq \frac{k(4 - n^{k-1}) + 4}{4k + 4} = \frac{2}{3}$$

which finishes the problem directly.

Generalization 2.
Let a, b, c be pozitive reals. Then prove that

$$\sum_{cyc} \frac{\left((k+1)a + kb + kc\right)^2}{(k+1)a^2 + kb^2 + kc^2} \leq 6k + \frac{3}{2k+1} + 1$$

Solution.
Let us see that the inequality is homogenous.WLOG Assume that $a + b + c = 1$.

$$\sum_{cyc} \frac{\left((k+1)a + kb + kc\right)^2}{(k+1)a^2 + kb^2 + kc^2} = \sum_{cyc} \frac{(a+k)^2}{(k+1)a^2 + k(1-a)^2} = \sum_{cyc} \frac{(a+k)^2}{(2k+1)a^2 - 2ak + k}$$

$$= \sum_{cyc} \left(\frac{1}{2k+1} + \frac{2ak + \frac{2ak}{2k+1} + k^2 - \frac{k}{2k+1}}{(2k+1)a^2 - 2ak + k} \right) = S$$

By making some changes on denominator.

$$(2k+1)a^2 - 2ak + k = (2k+1)a^2 - 2ak + \frac{k^2}{2k+1} + k - \frac{k^2}{2k+1} \overbrace{\geq}^{AGO} 2ak - 2ak + k - \frac{k^2}{2k+1} = k - \frac{k^2}{2k+1}$$

$$S \leq \sum_{cyc} \left(\frac{1}{2k+1} + \frac{2ak + \frac{2ak}{2k+1} + k^2 - \frac{k}{2k+1}}{k - \frac{k^2}{2k+1}} \right)$$

$$= \frac{3}{2k+1} + \frac{2k(a+b+c) + \frac{2k}{2k+1}(a+b+c) + 3k^2 - \frac{3k}{2k+1}}{k - \frac{k^2}{2k+1}}$$

$$= \frac{3}{2k+1} + \frac{2k + \frac{2k}{2k+1} + 3k^2 - \frac{3k}{2k+1}}{k - \frac{k^2}{2k+1}} = \frac{3}{2k+1} + \frac{(2k+1)\left(3k^2 + 2k - \frac{k}{2k+1}\right)}{k^2+k}$$

$$= \frac{3}{2k+1} + \frac{(2k+1)k(3k+2-\frac{1}{2k+1}}{k(k+1)} = \frac{3}{2k+1} + \frac{(2k+1)(3k+2-\frac{1}{2k+1})}{k+1}$$

$$= \frac{3}{2k+1} + \frac{(2k+1)\left(\frac{6k^2+7k+1}{2k+1}\right)}{k+1}$$

$$= \frac{3}{2k+1} + \frac{6k^2+7k+1}{k+1} = \frac{3}{2k+1} + 6k + 1$$

Problem has been solved.

Application on USAMO 2003 5

Let a, b, c be positive reals. Then prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8$$

Solution.

Problem is specialised use of *Generalization 2.* which $k = 1$. By replacing it

$$\sum_{cyc} \frac{((k+1)a+kb+kc)^2}{(k+1)a^2+kb^2+kc^2} \le 6k + \frac{3}{2k+1} + 1 = 8$$

which is the conclusion of the problem.

REFERENCES.

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