Weighted Nesbitt's inequality

Dorin Marghidanu ,
d.marghidanu@gmail.com

In this short note, a weighted version is presented and at in the same time an extension of Nesbitt's inequality. Consequences of this inequality are also presented

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It is known in mathematical practice and in mathematical literature - *Nesbitt's* famous and beautiful *inequality* , [1] :

- if $a , b , c > 0$, then , $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \quad , \qquad \textbf{(N)}$$

For this famous inequality, there are dozens of proofs , extensions , generalizations and various refinements . Our intention is to obtain an inequality, when in the left member of inequality (N) weights appear. For this we will use *Jensen's weighted inequality* :

- if $f : \mathrm{I} \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function , I – interval , then for any $x_k \in \mathbf{I}$

and any weights $w_k > 0$, $k \in \{1, 2, \cdots, n\}$, for which we have $\sum_{k=1}^{n} w_k x_k \in \mathbf{I}$, $\sum_{k=1}^{n} w_k = 1$,

we have the inequality $$\sum_{k=1}^{n} w_k f(x_k) \geq f\left(\sum_{k=1}^{n} w_k x_k\right) \quad , \qquad \textbf{(J)}$$

with equality if and only if $x_1 = x_2 = ... = x_n$.

We will thus have the following statement ,

1. *Proposition* **(*weighted Nesbitt's inequality*)**

For any $a , b , c > 0$ and any weights $m , n , p > 0$ with $m + n + p = 1$ holds the inequality ,

$$m \cdot \frac{a}{b+c} + n \cdot \frac{b}{c+a} + p \cdot \frac{c}{a+b} \geq \frac{ma + nb + pc}{(n+p)a + (p+m)b + (m+n)c} \quad , \qquad \textbf{(wN)}$$

with equality if and only if $a = b = c$.

Proof

With the notation $\mathrm{S} := a + b + c$ and remarking that $\frac{a}{b+c} = \frac{a}{S-a}$, etc.

let be the function $f : (0 , \mathrm{S}) \longrightarrow \mathbb{R}$, $f(x) = \frac{x}{S-x}$, for which we have

$$f'(x) = \frac{S}{(S-x)^2} \quad , \quad f''(x) = \frac{2S}{(S-x)^3} \geq 0 ,$$ so the function is convex .

After a slight preparation , and then with an application of *Jensen's weighted inequality* for case $n=3$ and weights $m , n , p > 0$, we have :

$$m\cdot\frac{a}{b+c} + n\cdot\frac{b}{c+a} + p\cdot\frac{c}{a+b} = m\cdot f(a) + n\cdot f(b) + p\cdot f(c) \overset{Jensen}{\geq}$$

$$\overset{Jensen}{\geq} f(ma+nb+pc) = \frac{ma+nb+pc}{S-(ma+nb+pc)} = \frac{ma+nb+pc}{(1-m)a+(1-n)b+(1-p)c} =$$

$$= \frac{ma+nb+pc}{(n+p)a+(p+m)b+(m+n)c} \quad .$$ □

2. *Remark*

Taking $m=n=p\ (=1/3)$ in *Nesbitt's weighted inequality* **(wN)** we get *Nesbitt's classical inequality* (N) .

We exemplify with the following simple application ,

3. *Corolar*

For any $a , b , c > 0$ holds the inequality ,

$$2\cdot\frac{a}{b+c} + 3\cdot\frac{b}{c+a} + 4\cdot\frac{c}{a+b} \geq \frac{2a+3b+4c}{7a+6b+5c} \ ,$$

with equality if and only if $a = b = c$.

Proof

Choosing in *Nesbitt's weighted inequality* **(wN)** the weights , $m=2/9$, $n=3/9$, $p=4/9$, for which we obviously have $m+n+p=1$, we obtain the inequality from corollary .

4. *Proposition* ***(generalization of the weighted Nesbitt's inequality)*** **, [4], b)**

For any $a_1 , a_2 , . . . , a_n > 0$ and any weights $w_1 , w_2 , . . . , w_n > 0$, with $w_1 + w_2 + . . . + w_n = 1$, holds the inequality ,

$$w_1\cdot\frac{a_1}{a_2+a_3+\ldots+a_n} + w_2\cdot\frac{a_2}{a_1+a_3+\ldots+a_n} + \ldots + w_n\cdot\frac{a_n}{a_1+a_2+\ldots+a_{n-1}} \geq$$

$$\geq \frac{w_1a_1+w_2a_2+\ldots+w_na_n}{(1-w_1)a_1+(1-w_2)a_2+\ldots+(1-w_n)a_n} \quad . \qquad \textbf{(gwN)}$$

with equality if and only if , $a_1 = a_2 = \ . . . = a_n$.

Proof

With the notation $\mathbf{S} := a_1 + a_2 + . . . + a_n$, we also consider here the function

$f:(0,S)\longrightarrow\mathbb{R}$, $f(x)=\dfrac{x}{S-x}$, which (as we saw in the proof of *Proposition* 1) is a *convex function* on $(0,S)$.

After an easy preparation , and then with the application of *Jensen's weighted inequality* (J) , we have :

$$w_1\cdot\frac{a_1}{a_2+a_3+\ldots+a_n}+w_2\cdot\frac{a_2}{a_1+a_3+\ldots+a_n}+\ldots+w_n\cdot\frac{a_n}{a_1+a_2+\ldots+a_{n-1}}=$$

$$=w_1\cdot\frac{a_1}{S-a_1}+w_2\cdot\frac{a_2}{S-a_2}+\ldots+w_n\cdot\frac{a_n}{S-a_n}=$$

$$=w_1\cdot f(a_1)+w_2\cdot f(a_2)+\ldots+w_n\cdot f(a_n)\overset{Jensen}{\geq}$$

$$\overset{Jensen}{\geq} f(w_1a_1+w_2a_2+\ldots+w_na_n)=\frac{w_1a_1+w_2a_2+\ldots+w_na_n}{S-(w_1a_1+w_2a_2+\ldots+w_na_n)}=$$

$$=\frac{w_1a_1+w_2a_2+\ldots+w_na_n}{(1-w_1)a_1+(1-w_2)a_2+\ldots+(1-w_n)a_n}.$$

5. *Remark*

Taking $w_1=w_2=\ldots=w_n=1/n$ in the generalization of *Nesbitt's weighted inequality* **(gwN)** the *generalization of Nesbitt's classical inequality* (gN) is obtained :

$$\frac{a_1}{a_2+a_3+\ldots+a_n}+\frac{a_2}{a_1+a_3+\ldots+a_n}+\ldots+\frac{a_n}{a_1+a_2+\ldots+a_{n-1}}\geq\frac{n}{n-1}. \qquad \text{(gN)}$$

(Olympiad , German Democratic Republic ,1967)..

For $n=3$, the *classical Nesbitt's inequality* (N) is obtained .

By customizing the weights in *Proposition* 4 , numerous inequalities can be obtained . Here is an example :

6. *Corolar* , [4], a)

For any $a,b,c,d>0$, holds the inequality ,

$$\frac{a}{b+c+d}+2\cdot\frac{b}{c+d+a}+3\cdot\frac{c}{d+a+b}+4\cdot\frac{d}{a+b+c}\geq 10\cdot\frac{a+2b+3c+4d}{9a+8b+7c+6d},$$

with equality if and only if $a=b=c=d$.

Proof

Taking in **(gwN)** , $n=4$, the weights : $w_1=1/10$, $w_2=2/10$, $w_3=3/10$, $w_4=4/10$, for which we obviously have $w_1+w_2+w_3+w_4=1$, the inequality from the statement is obtained .

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