A SIMPLE PROOF FOR JANIC & VASIC'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper we will give a simple proof for Janic & Vasic's inequality and a few applications.

JANIC & VASIC'S INEQUALITY

If $0 < k \leq n; n \in \mathbb{N}^*; a_1, a_2, \ldots, a_n > 0$ then:

$$\sum_{k=1}^{n} \frac{a_1 + a_2 + \ldots + a_k}{a_{k+1} + a_{k+2} + \ldots + a_n} \geq \frac{nk}{n-k} \tag{1}$$

Proof.

$$\sum_{k=1}^{n} \frac{a_1 + a_2 + \ldots + a_k}{a_{k+1} + a_{k+2} + \ldots + a_n} = \sum_{k=1}^{n} \Big(\frac{a_1 + a_2 + \ldots + a_k}{a_{k+1} + a_{k+2} + \ldots + a_n} + 1 \Big) - n =$$

$$= \sum_{k=1}^{n} \frac{a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots + a_n}{a_{k+1} + a_{k+2} + \ldots + a_n} - n =$$

$$= (a_1 + a_2 + \ldots + a_n) \sum_{k=1}^{n} \frac{1}{a_{k+1} + a_{k+2} + \ldots + a_n} - n \geq$$

$$\overset{\text{BERGSTRÖM}}{\geq} (a_1 + a_2 + \ldots + a_n) \cdot \frac{(\underbrace{1 + 1 + \ldots + 1}_{\text{for "}n\text{" times}})^2}{(n-k)(a_1 + a_2 + \ldots + a_n)} - n =$$

$$= \frac{n^2}{n-k} - n = \frac{n^2 - n^2 + nk}{n-k} = \frac{nk}{n-k}$$

□

Application 1.

If $a, b, c > 0$ then:

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6 \tag{2}$$

Proof.

We take in $(1) : n = 3; k = 2$

Equality holds for $a = b = c$. □

Application 2.

If $x, y, z > 1$ then:

$$\log_z(xy) + \log_x(yz) + \log_y(zx) \geq 6$$

Proof.

$$\text{We take in (2)} : a = \ln x; b = \ln y; c = \ln z$$

$$\frac{\ln x + \ln y}{\ln z} + \frac{\ln y + \ln z}{\ln x} + \frac{\ln z + \ln x}{\ln y} \geq 6$$

$$\frac{\ln(xy)}{\ln z} + \frac{\ln(yz)}{\ln x} + \frac{\ln(zx)}{\ln y} \geq 6$$

$$\log_z(xy) + \log_x(yz) + \log_y(zx) \geq 6$$

Equality holds for: $x = y = z$. □

Application 3.
If $a, b, c, d > 0$ then:

$$\frac{a+b+c}{d} + \frac{b+c+d}{a} + \frac{c+d+a}{b} + \frac{d+a+b}{c} \geq 12 \tag{3}$$

Proof.

$$\text{We take in (1)} : n = 4; k = 3.$$

Equality holds for $a = b = c = d$. □

Application 4.
If $x, y, z, t > 1$ then:

$$\log_t(xyz) + \log_x(yzt) + \log_y(ztx) + \log_z(txy) \geq 12$$

Proof.

$$\text{We take in (3)}$$

$$a = \ln x; b = \ln y; c = \ln z; d = \ln t$$

$$\frac{\ln x + \ln y + \ln z}{\ln t} + \frac{\ln y + \ln z + \ln t}{\ln x} + \frac{\ln z + \ln t + \ln x}{\ln y} + \frac{\ln t + \ln x + \ln y}{\ln z} \geq 12$$

$$\frac{\ln(xyz)}{\ln t} + \frac{\ln(yzt)}{\ln x} + \frac{\ln(ztx)}{\ln y} + \frac{\ln(txy)}{\ln z} \geq 12$$

$$\log_t(xyz) + \log_x(yzt) + \log_y(ztx) + \log_z(txy) \geq 12$$

Equality holds for $x = y = z = t$. □

Application 5.
If $a, b, c, d, e > 0$ then:

$$\frac{a+b+c}{d+e} + \frac{b+c+d}{e+a} + \frac{c+d+a}{a+b} + \frac{d+a+b}{b+c} + \frac{a+b+c}{c+d} \geq \frac{15}{2} \tag{4}$$

Proof.

$$\text{We take in (1)} : n = 5; k = 3$$

□

Application 6.
If $x, y, z, t, u > 1$ then:

$$\log_{tu}(xyz) + \log_{ux}(yzt) + \log_{xy}(ztu) + \log_{yz}(tux) + \log_{zt}(uxy) \geq \frac{15}{2}$$

Proof.

$$\text{We take in (4) :}$$
$$a = \ln x; b = \ln y; x = \ln z; d = \ln t; e = \ln u$$
$$a, b, c, d, e > 0$$
$$\sum_{cyc} \frac{\ln x + \ln y + \ln z}{\ln t + \ln u} \geq \frac{15}{2}$$
$$\sum_{cyc} \frac{\ln(xyz)}{\ln(tu)} \geq \frac{15}{2}$$
$$\sum_{cyc} \log_{tu}(xyz) \geq \frac{15}{2}$$

Equality holds for $x = y = z = t = u$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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