A SIMPLE PROOF FOR ROGER'S INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper we will give a simple proof for Roger's inequality and a few applications.

ROGER'S INEQUALITY

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n > 0; n \in \mathbb{N}^*; s \in (0,1)$ then:

(1) $$\sum_{i=1}^{n} a_i^s \cdot b_i^{1-s} \leq \left(\sum_{i=1}^{n} a_i\right)^s \cdot \left(\sum_{i=1}^{n} b_i\right)^{1-s}$$

Proof.

$$\text{Let be } f:(0,\infty)\to\mathbb{R}; f(x) = x^\alpha - \alpha x; \alpha \in (0,1)$$

$$f'(x) = \alpha x^{\alpha-1} - \alpha = \alpha(x^{\alpha-1} - 1)$$

$$f'(x) = 0 \Rightarrow x^{\alpha-1} - 1 = 0 \Rightarrow x^{\alpha-1} = 1 \Rightarrow x = 1$$

$$f \text{ increasing on } (0,1)$$

$$f \text{ decreasing on } (1,\infty)$$

$$\max_{x>0} f(x) = f(1) = 1 - \alpha$$

(2) $$f(x) \leq f(1) \Rightarrow x^\alpha - \alpha x \leq 1 - \alpha$$

$$\text{Let be } p = \frac{1}{\alpha}; q = \frac{p}{p-1} = \frac{1}{1-\alpha}$$

$$\frac{1}{p} + \frac{1}{q} = \alpha + 1 - \alpha = 1$$

$$\alpha \in (0,1) \Rightarrow p > 1$$

$$\text{For } A, B > 0 \text{ let be } x = \frac{A}{B} \text{ in } (2):$$

$$\left(\frac{A}{B}\right)^\alpha - \alpha\left(\frac{A}{B}\right) \leq 1 - \alpha$$

$$\left(\frac{A}{B}\right)^{\frac{1}{p}} - \frac{1}{p}\cdot\left(\frac{A}{B}\right) \leq 1 - \frac{1}{p} = \frac{1}{q}$$

$$\left(\frac{A}{B}\right)^{\frac{1}{p}} - \frac{1}{p}\left(\frac{A}{B}\right) \leq \frac{1}{q}$$

$$\frac{A^{\frac{1}{p}}}{B^{\frac{1}{p}}} - \frac{A}{pB} \leq \frac{1}{q}$$

$$\frac{A^{\frac{1}{p}} \cdot B^{\frac{1}{q}}}{B^{\frac{1}{p}}} - \frac{A}{p} \cdot B^{\frac{1}{q}-1} \leq \frac{1}{q} \cdot B^{\frac{1}{q}}$$

$$A^{\frac{1}{p}} \cdot B^{\frac{1}{q}} - \frac{A}{p} \cdot B^{\frac{1}{p}+\frac{1}{q}-1} \leq \frac{1}{q} \cdot B^{\frac{1}{p}+\frac{1}{q}}$$

$$A^{\frac{1}{p}} \cdot B^{\frac{1}{q}} - \frac{A}{p} \cdot B^{1-1} \leq \frac{1}{q} \cdot B$$

$$A^{\frac{1}{p}} \cdot B^{\frac{1}{q}} \leq \frac{A}{p} + \frac{B}{q} \tag{3}$$

Let be:

$$A = \frac{x_i^p}{\sum_{i=1}^n x_i^p}; B = \frac{y_i^q}{\sum_{i=1}^n y_i^q} \text{ in } (3)$$

$$\left(\frac{x_i^p}{\sum_{i=1}^n x_i^p}\right)^{\frac{1}{p}} \cdot \left(\frac{y_i^q}{\sum_{i=1}^n y_i^q}\right)^{\frac{1}{q}} \leq \frac{1}{p} \cdot \frac{x_i^p}{\sum_{i=1}^n x_i^p} + \frac{1}{q} \cdot \frac{y_i^q}{\sum_{i=1}^n y_i^2}$$

$$\frac{x_i \cdot y_i}{\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}}} \leq \frac{1}{p\sum_{i=1}^n x_i^p} \cdot x_i^p + \frac{1}{q \cdot \sum_{i=1}^n y_i^q} \cdot y_i^q$$

By summing:

$$\frac{\sum_{i=1}^n x_i y_i}{\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}}} \leq \frac{1}{p} \cdot \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n x_i^p} + \frac{1}{q} \cdot \frac{\sum_{i=1}^n y_i^2}{\sum_{i=1}^n y_i^q} =$$

$$= \frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = \frac{1}{p} + \frac{1}{q} = 1$$

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \text{ (Hölder)}$$

In Hölder's inequality we take:

$$x_i = a_i^s; y_i = b_i^{1-s}$$

$$p = \frac{1}{s}; q = \frac{1}{1-s}$$

$$\sum_{i=1}^n a_i^s b_i^{1-s} \leq \left(\sum_{i=1}^n (a_i^s)^{\frac{1}{s}}\right)^s \cdot \left(\sum_{i=1}^n (b_i^{\frac{1}{1-s}})^{\frac{1}{1-s}}\right)^{1-s}$$

$$\sum_{i=1}^n a_i^s b_i^{1-s} \leq \left(\sum_{i=1}^n a_i\right)^s \cdot \left(\sum_{i=1}^n b_i\right)^{1-s}$$

□

Application 1 (Huygens' inequality)
If $a_1, a_2, b_1, b_2 > 0$ then:

$$(a_1 + a_2)(b_1 + b_2) \geq (\sqrt{a_1 b_1} + \sqrt{a_2 b_2})^2$$

Proof.

$$\text{We take in (1)} : n = 2; s = \frac{1}{2}$$

$$a_1^{\frac{1}{2}} \cdot b_1^{\frac{1}{2}} + a_2^{\frac{1}{2}} \cdot b_2^{\frac{1}{2}} \leq (a_1 + a_2)^{\frac{1}{2}} (b_1 + b_2)^{1-\frac{1}{2}}$$

$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} \leq \sqrt{(a_1 + a_2)(b_1 + b_2)}$$

$$(a_1 + a_2)(b_1 + b_2) \geq (\sqrt{a_1 b_1} + \sqrt{a_2 b_2})$$

□

Application 2.
If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \geq (\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3})^2$$

Proof.

$$\text{We take in (1) : } n = 3; s = \frac{1}{2}$$

$$a_1^{\frac{1}{2}} \cdot b_1^{\frac{1}{2}} + a_2^{\frac{1}{2}} \cdot b_2^{\frac{1}{2}} + a_3^{\frac{1}{2}} \cdot b_3^{\frac{1}{2}} \leq (a_1 + a_2 + a_3)^{\frac{1}{2}} \cdot (b_1 + b_2 + b_3)^{1-\frac{1}{2}}$$

$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3} \leq \sqrt{(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)}$$

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \geq (\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3})^2$$

□

Application 3.
If $a_1, a_2, b_1, b_2 > 0$ then:

$$(a_1 + a_2)(b_1 + b_2)^2 \geq (\sqrt[3]{a_1 b_1^2} + \sqrt[3]{a_2 b_2^2})^3$$

Proof.

$$\text{We take in (1) : } n = 2; s = \frac{1}{3}$$

$$a_1^{\frac{1}{3}} \cdot b_1^{\frac{2}{3}} + a_2^{\frac{1}{3}} \cdot b_2^{\frac{2}{3}} \leq (a_1 + a_2)^{\frac{1}{3}} \cdot (b_1 + b_2)^{\frac{2}{3}}$$

$$\sqrt[3]{a_1 b_1^2} + \sqrt[3]{a_2 b_2^2} \leq \sqrt[3]{(a_1 + a_2)(b_1 + b_2)^2}$$

$$(a_1 + a_2)(b_1 + b_2)^2 \geq (\sqrt[3]{a_1 b_1^2} + \sqrt[3]{a_2 b_2^2})^3$$

□

Application 4.
If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)^2 \geq (\sqrt[3]{a_1 b_1^2} + \sqrt[3]{a_2 b_2^2} + \sqrt[3]{a_3 b_3^2})^3$$

Proof.

$$\text{We take in (1) : } n = 3; s = \frac{1}{3}$$

$$a_1^{\frac{1}{3}} b_1^{\frac{2}{3}} + a_2^{\frac{1}{3}} b_2^{\frac{2}{3}} + a_3^{\frac{1}{3}} \cdot b_3^{\frac{2}{3}} \leq (a_1 + a_2 + a_3)^{\frac{1}{3}} \cdot (b_1 + b_2 + b_3)^{\frac{2}{3}}$$

$$\sqrt[3]{a_1 b_1^2} + \sqrt[3]{a_2 b_2^2} + \sqrt[3]{a_3 b_3^2} \leq \sqrt[3]{(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)^2}$$

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)^2 \geq (\sqrt[3]{a_1 b_1^2} + \sqrt[3]{a_2 b_2^2} + \sqrt[3]{a_3 b_3^2})^3$$

□

Application 5.
If $a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_n > 0; n \in \mathbb{N}^*; s \in (0, 1)$ then:

$$\sum_{i=1}^{n} a_i^s \cdot \left(\frac{1}{b_i}\right)^{1-s} \leq \left(\sum_{i=1}^{n} a_i\right)^s \cdot \left(\sum_{i=1}^{n} \frac{1}{b_i}\right)^{1-s} \tag{4}$$

Proof.

$$\text{We replace in } (1): b_i \to \frac{1}{b_i}$$

□

Application 6.
If $a_1, a_2, b_1, b_2 > 0$ then:

$$(a_1+a_2)\Big(\frac{1}{b_1}+\frac{1}{b_2}\Big) \geq \Big(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}\Big)^2$$

Proof.

$$\text{We take in } (4): n=2; s=\frac{1}{2}$$

$$a_1^{\frac{1}{2}}\cdot\Big(\frac{1}{b_1}\Big)^{1-\frac{1}{2}}+a_2^{\frac{1}{2}}\cdot\Big(\frac{1}{b_2}\Big)^{1-\frac{1}{2}} \leq (a_1+a_2)^{\frac{1}{2}}\cdot\Big(\frac{1}{b_1}+\frac{1}{b_2}\Big)^{1-\frac{1}{2}}$$

$$\frac{a_1^{\frac{1}{2}}}{b_1^{\frac{1}{2}}}+\frac{a_2^{\frac{1}{2}}}{b_2^{\frac{1}{2}}} \leq (a_1+a_2)^{\frac{1}{2}}\cdot\Big(\frac{1}{b_1}+\frac{1}{b_2}\Big)^{\frac{1}{2}}$$

$$\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}} \leq \sqrt{(a_1+a_2)\Big(\frac{1}{b_1}+\frac{1}{b_2}\Big)}$$

$$(a_1+a_2)\Big(\frac{1}{b_1}+\frac{1}{b_2}\Big) \geq \Big(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}\Big)^2$$

□

Application 7.
If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1+a_2+a_3)\Big(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\Big) \geq \Big(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}+\sqrt{\frac{a_3}{b_3}}\Big)^2$$

Proof.

$$\text{We take in } (4): n=3; s=\frac{1}{2}$$

$$a_1^{\frac{1}{2}}\cdot\Big(\frac{1}{b_1}\Big)^{1-\frac{1}{2}}+a_2^{\frac{1}{2}}\cdot\Big(\frac{1}{b_2}\Big)^{1-\frac{1}{2}}+a_3^{\frac{1}{2}}\cdot\Big(\frac{1}{b_3}\Big)^{1-\frac{1}{2}} \leq (a_1+a_2+a_3)^{\frac{1}{2}}\Big(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\Big)^{1-\frac{1}{2}}$$

$$\frac{a_1^{\frac{1}{2}}}{b_1^{\frac{1}{2}}}+\frac{a_2^{\frac{1}{2}}}{b_2^{\frac{1}{2}}}+\frac{a_3^{\frac{1}{2}}}{b_3^{\frac{1}{2}}} \leq (a_1+a_2+a_3)^{\frac{1}{2}}\cdot\Big(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\Big)^{\frac{1}{2}}$$

$$\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}+\sqrt{\frac{a_3}{b_3}} \leq \sqrt{(a_1+a_2+a_3)\Big(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\Big)}$$

$$(a_1+a_2+a_3)\Big(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\Big) \geq \Big(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}+\sqrt{\frac{a_3}{b_3}}\Big)^2$$

□

Application 8.
If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1+a_2+a_3)\Big(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\Big)^2 \geq \Big(\sqrt[3]{\frac{a_1}{b_1^2}}+\sqrt[3]{\frac{a_2}{b_2^2}}+\sqrt[3]{\frac{a_3}{b_3^2}}\Big)^3$$

Proof.

$$\text{We take in (4)} : n = 3; s = \frac{1}{3}$$

$$a_1^{\frac{1}{3}} \cdot \Big(\frac{1}{b_1}\Big)^{1-\frac{1}{3}} + a_2^{\frac{1}{3}} \cdot \Big(\frac{1}{b_2}\Big)^{1-\frac{1}{3}} + a_3^{\frac{1}{3}} \cdot \Big(\frac{1}{b_3}\Big)^{\frac{1}{3}} \leq (a_1 + a_2 + a_3)^{\frac{1}{3}} \Big(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}\Big)^{1-\frac{1}{3}}$$

$$\sqrt[3]{\frac{a_1}{b_1^2}} + \sqrt[3]{\frac{a_2}{b_2^2}} + \sqrt[3]{\frac{a_3}{b_3^2}} \leq \sqrt[3]{(a_1 + a_2 + a_3)\Big(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}\Big)^2}$$

$$(a_1 + a_2 + a_3)\Big(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}\Big)^2 \geq (\sqrt[3]{\frac{a_1}{b_1}} + \sqrt[3]{\frac{a_2}{b_2}} + \sqrt[3]{\frac{a_3}{b_3}}\Big)^3$$

□

Application 9.
If $x, y \in (0, \frac{\pi}{2})$ then:

$$\sqrt{\sin 2x} + \sqrt{\sin 2y} \leq \sqrt{2(\sin x + \sin y)(\cos x + \cos y)}$$

Proof.

$$\text{We take in (1)} : n = 2; s = \frac{1}{2};$$

$$a_1 = \sin x; a_2 = \sin y; b_1 = \cos x; b_2 = \cos y$$

$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} \leq \sqrt{(a_1 + a_2)(b_1 + b_2)}$$

$$\sqrt{\sin x \cos x} + \sqrt{\sin y \cos y} \leq \sqrt{(\sin x + \sin y)(\cos x + \cos y)}$$

$$\sqrt{2 \sin x \cos x} + \sqrt{2 \sin y \cos y} \leq \sqrt{2(\sin x + \sin y)(\cos x + \cos y)}$$

$$\sqrt{\sin 2x} + \sqrt{\sin 2y} \leq \sqrt{2(\sin x + \sin y)(\cos x + \cos y)}$$

□

Application 10.
If $x, y \in (0, \frac{\pi}{2})$ then:

$$\sqrt{\sin 2x} + \sqrt{\sin 2y} + \sqrt{\sin 2z} \leq \sqrt{2(\sin x + \sin y + \sin z)(\cos x + \cos y + \cos z)}$$

Proof.

$$\text{We take in (1)} : n = 3; s = \frac{1}{2}$$

$$a_1 = \sin x; a_2 = \sin y; a_3 = \sin z$$

$$b_1 = \cos x; b_2 = \cos y; b_3 = \cos z$$

$$\sqrt{\sin x \cos x} + \sqrt{\sin y \cos y} + \sqrt{\sin z \cos z} \leq \sqrt{(\sin x + \sin y + \sin z)(\cos x + \cos y + \cos z)}$$

$$\sqrt{2 \sin x \cos x} + \sqrt{2 \sin x \cos y} + \sqrt{2 \sin z \cos z} \leq \sqrt{2(\sin x + \sin y + \sin z)(\cos x + \cos y + \cos z)}$$

$$\sqrt{\sin 2x} + \sqrt{\sin 2y} + \sqrt{\sin 2z} \leq \sqrt{2(\sin x + \sin y + \sin z)(\cos x + \cos y + \cos z)}$$

□

Application 11.
If $x, y, z, t \in \mathbb{R}$ then:

$$\sqrt{e^{x+y}} + \sqrt{e^{z+t}} \leq \sqrt{(e^x + e^z)(e^y + e^t)}$$

Proof.

$$\text{We take in } (1): n = 2; s = \frac{1}{2}$$
$$a_1 = e^x; a_2 = e^z$$
$$b_1 = e^y; b_2 = e^t$$
$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} \leq \sqrt{(a_1 + a_2)(b_1 + b_2)}$$
$$\sqrt{e^x \cdot e^y} + \sqrt{e^z \cdot e^t} \leq \sqrt{(e^x + e^z)(e^y + e^t)}$$
$$\sqrt{e^{x+y}} + \sqrt{e^{z+t}} \leq \sqrt{(e^x + e^z)(e^y + e^t)}$$

□

References

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