CERTAIN LIMITS OF FIBONACCI AND LUCAS SEQUENCES AND FUNCTIONS

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Abstract. In this paper we present new limits of sequences and functions.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$.
Lucas sequence: $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$.

Theorem 1.

$$\lim_{n\to\infty} (\sqrt[n]{n!})^2 \Big(\frac{\sqrt[n]{n!L_n}}{n^2} - \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2} \Big) = \frac{\alpha}{e^3} \Big(1 + \frac{1}{2} \ln 5 \Big)$$

Proof.

$$\text{We have } L_n = \alpha^n + \beta^n, F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!L_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!L_n}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!L_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!L_n} =$$

$$= \lim_{n\to\infty} \Big(\frac{n}{n+1} \Big) \frac{L_{n+1}}{L_n} = \frac{1}{e} \cdot \alpha = \frac{\alpha}{e}, \text{ and analogous } \lim_{n\to\infty} \frac{\sqrt[n]{n!F_n}}{n} = \frac{\alpha}{e}.$$

$$\text{If we denote } u_n = \frac{\sqrt[n]{n!L_n}}{\sqrt[n+1]{(n+1)!F_{n+1}}} \Big(\frac{n+1}{n} \Big)^2, \text{ we deduce}$$

$$\lim_{n\to\infty} u_n = 1; \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1 \text{ and}$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{n!L_n}{(n+1)!F_{n+1}} \Big(\frac{n+1}{n} \Big)^{2n} \sqrt[n+1]{(n+1)!F_{n+1}} =$$

$$= e^2 \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{n+1} \cdot \frac{L_n}{F_{n+1}} =$$

$$= e^2 \cdot \frac{\alpha}{e} \cdot \lim_{n\to\infty} \frac{\sqrt{5}(\alpha^n + \beta^n)}{\alpha^{n+1} - \beta^{n+1}} = \alpha e \cdot \frac{\sqrt{5}}{\alpha} = e\sqrt{5}$$

Hence,

$$\lim_{n\to\infty} (\sqrt[n]{n!})^2 \Big(\frac{\sqrt[n]{n!L_n}}{n^2} - \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2} \Big) = \lim_{n\to\infty} (\sqrt[n]{n!})^2 \cdot \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2} (u_n - 1) =$$

$$= \lim_{n\to\infty} \frac{(\sqrt[n]{n!})^2 \sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \lim_{n\to\infty} \Big(\frac{\sqrt[n]{n!}}{n} \Big)^2 \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{n+1} \cdot \frac{n}{n+1} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n =$$

$$= \frac{1}{e^2} \cdot \frac{\alpha}{e} \cdot 1 \cdot 1 \cdot \ln(e\sqrt{5}) = \frac{\alpha}{e^3}\left(1 + \frac{1}{2}\ln 5\right)$$

□

Theorem 2.
If $m \in \mathbb{N}$, then:

$$\lim_{n\to\infty} n^{\cos^2 F_m}((\sqrt[n+1]{(n+1)!})^{\sin^2 F_m} - (\sqrt[n]{n!})^{\sin^2 F_m}) = \frac{\sin^2 F}{e^{\sin^2 F_m}}$$

Proof.

We have well-known $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$, so $\lim_{n\to\infty} \left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m} = e^{-\sin^2 F_m}$.

We denote $u_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{\sin^2 F_m}, \forall n \geq 2$, so $\lim_{n\to\infty} u_n = 1$ and $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$.

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^{\sin^2 F_m} \frac{1}{(\sqrt[n+1]{(n+1)!})^{\sin^2 F_m}} =$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{\sin^2 F_m} = e^{\sin^2 F_m}$$

We have $x_n \overset{\text{denote}}{=} n^{\cos^2 F_m}((\sqrt[n+1]{(n+1)!})^{\sin^2 F_m} - (\sqrt[n]{n!})^{\sin^2 F_m}) =$

$$= n^{\cos^2 F_m}(u_n - 1)(\sqrt[n]{n!})^{\sin^2 F_m} =$$

$$= n^{\cos^2 F_m + \sin^2 F_m}\left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m}(u_n - 1) = n\left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m} \frac{u_n - 1}{\ln u_n} \ln u_n =$$

$$= \left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m} \frac{u_n - 1}{\ln u_n} \ln u_n^n.$$

Hence,

$$\lim_{n\to\infty} n^{\cos^2 F_m}((\sqrt[n+1]{(n+1)!})^{\sin^2 F_m} - (\sqrt[n]{n!})^{\sin^2 F_m}) =$$

$$= \lim_{n\to\infty} x_n = e^{-\sin^2 F_m} \cdot 1 \cdot \ln e^{\sin^2 F_m} = \frac{\sin^2 F_m}{e^{\sin^2 F_m}}$$

□

Theorem 3.

$$\lim_{n\to\infty} \left(\int_{\sqrt[n]{n!F_n}}^{\sqrt[n+1]{(n+1)!F_{n+1}}} \sqrt{\frac{\sin^4 x + \cos^4 x}{8}} dx\right) \geq \frac{\alpha}{4e}.$$

Proof.
By Bergström's inequality we have

$$\sin^4 x + \cos^4 x \geq \frac{(\sin^2 x + \cos^2 x)^2}{2} = \frac{1}{2} \tag{1}$$

From (1) we deduce that

$$I_n = \int_{\sqrt[n]{n!F_n}}^{\sqrt[n+1]{(n+1)!F_{n+1}}} \sqrt{\frac{\sin^4 x + \cos^4 x}{8}} dx \geq$$

$$\geq \frac{1}{4} \int_{\sqrt[n]{n!F_n}}^{\sqrt[n+1]{(n+1)!F_{n+1}}} dx = \frac{1}{4}(\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n}) \tag{2}$$

Next, we have:

$$F_{n+2} - F_{n+1} - F_n = 0, \forall n \in \mathbb{N}^* \Leftrightarrow \frac{F_{n+2}}{F_n} - \frac{F_{n+1}}{F_n} - 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{F_{n+2}}{F_{n+1}} \cdot \frac{F_{n+1}}{F_n} - \frac{F_{n+1}}{F_n} - 1 = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{F_{n+2}}{F_{n+1}} \cdot \frac{F_{n+1}}{F_n} - \frac{F_{n+1}}{F_n} - 1\right) = 0 \Leftrightarrow x^2 - x - 1 = 0, \text{ where } x = \lim_{n\to\infty}\frac{F_{n+1}}{F_n}$$

$$\text{so } x = \frac{1 \pm \sqrt{5}}{2}. \text{ Since } \frac{F_{n+1}}{F_n} > 0, \text{ yields } x = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2} = \alpha.$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!F_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!F_n}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!F_n} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{\alpha}{e}.$$

Therefore

$$\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n} = \sqrt[n]{n!F_n}(u_n - 1) = \sqrt[n]{n!F_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \frac{\sqrt[n]{n!F_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2 \tag{3}$$

$$\text{where } u_n = \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{\sqrt[n]{n!F_n}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{n!F_n}} \cdot \frac{n+1}{n}\right) = \frac{\alpha}{e} \cdot \frac{e}{\alpha} \cdot 1 = 1, \text{ so}$$

$$\lim_{n\to\infty}\frac{u_n - 1}{\ln u_n} = 1, \text{ and}$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{n!F_n} \cdot \frac{1}{\sqrt[n+1]{(n+1)F_{n+1}}} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!F_{n+1}}} =$$

$$= \alpha \cdot \frac{e}{\alpha} = e.$$

By (3) we obtain

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n}\right) = \frac{\alpha}{e} \cdot 1 \cdot \ln(\lim_{n\to\infty} u_n^n) = \frac{\alpha}{e} \cdot \ln e = \frac{\alpha}{e} \cdot 1 = \frac{\alpha}{e} \tag{4}$$

From (2) and (4) yields the desired conclusion. □

Theorem 4.

$$\lim_{n\to\infty}\left(\int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}} \sqrt[3]{\frac{\sin^6 x + \cos^6 x}{16}}dx\right) \geq \frac{\alpha}{4e}.$$

Proof.
By Radon's inequality we have

$$\sin^6 x + \cos^6 x \geq \frac{(\sin^2 x + \cos^2 x)^3}{2^2} = \frac{1}{4} \tag{1}$$

From (1) we deduce that

$$I_n = \int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}} \sqrt[3]{\frac{\sin^6 x + \cos^6 x}{16}}dx\Bigg) \geq \int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}} \sqrt[3]{\frac{1}{4^3}}dx =$$

$$(2) \qquad = \frac{1}{4}\int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}} dx = \frac{1}{4}\left(\sqrt[n+1]{(n+1)!L_{n+1}} - \sqrt[n]{n!L_n}\right)$$

Now,

$$L_{n+2}-L_{n+1}-L_n = 0, \forall n \in \mathbb{N}^* \Leftrightarrow \frac{L_{n+2}}{L_n} - \frac{L_{n+1}}{L_n} - 1 = 0 \Leftrightarrow \frac{L_{n+2}}{L_{n+1}} \cdot \frac{L_{n+1}}{L_n} - \frac{L_{n+1}}{L_n} - 1 = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{L_{n+2}}{L_{n+1}} \cdot \frac{L_{n+1}}{L_n} - \frac{L_{n+1}}{L_n} - 1\right) = 0 \Leftrightarrow x^2 - x - 1 = 0, \text{ where } x = \lim_{n\to\infty}\frac{L_{n+1}}{L_n}, \text{ so}$$

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

$$\text{Since } \frac{L_{n+1}}{L_n} > 0, \text{ yields } x = \lim_{n\to\infty}\frac{L_{n+1}}{L_n} = \frac{\sqrt{5}+1}{2} = \alpha.$$

We deduce

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!L_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!L_n}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!L_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!L_n} = \lim_{n\to\infty}\frac{L_{n+1}}{L_n}\left(\frac{n}{n+1}\right)^n = \frac{\alpha}{e}.$$

We have

$$\sqrt[n+1]{(n+1)!L_{n+1}} - \sqrt[n]{n!L_n} = \sqrt[n]{n!L_n}(u_n - 1) = \sqrt[n]{n!L_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$(3) \qquad = \frac{\sqrt[n]{n!L_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2$$

$$\text{where } u_n = \frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{\sqrt[n]{n!L_n}}.$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{n!L_n}} \cdot \frac{n+1}{n}\right) = \frac{\alpha}{e} \cdot \frac{e}{\alpha} \cdot 1 = 1$$

$$\text{so } \lim_{n\to\infty}\frac{u_n - 1}{\ln u_n} = 1, \text{ and}$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{(n+1)!L_{n+1}}{n!L_n} \cdot \frac{1}{\sqrt[n+1]{(n+1)!L_{n+1}}} = \lim_{n\to\infty}\frac{L_{n+1}}{L_n} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!L_{n+1}}} =$$

$$= \alpha \cdot \frac{e}{\alpha} = e.$$

By (3) we obtain

$$(4) \quad \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!L_{n+1}} - \sqrt[n]{n!L_n}\right) = \frac{\alpha}{e} \cdot 1 \cdot \ln\left(\lim_{n\to\infty} u_n^n\right) = \frac{\alpha}{e} \cdot \ln e = \frac{\alpha}{e} \cdot 1 = \frac{\alpha}{e}$$

From (2) and (4) yields the desired conclusion. □

Theorem 5.

$$\text{If } a > 0, \text{ then } \lim_{n\to\infty}\sqrt[n]{n!F_n}\left(\sqrt[n]{a} - 1\right) = \frac{\alpha}{e}\ln a.$$

Proof.

$$\text{Since } \sqrt[n]{n!F_n}(\sqrt[n]{a}-1) = \sqrt[n]{n!F_n}\left(e^{\frac{\ln a}{n}}-1\right) = \sqrt[n]{n!F_n}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\frac{\ln a}{n} =$$

$$= \sqrt[n]{\frac{n!F_n}{n^n}}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\ln a, \text{ we have } \lim_{n\to\infty}\sqrt[n]{n!F_n}(\sqrt[n]{a}-1) = \ln a\cdot 1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n!F_n}{n^n}} =$$

$$= \ln a\cdot\lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!F_n} = \ln a\cdot\lim_{n\to\infty}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{\alpha}{e}\ln a$$

□

Theorem 6.

$$\text{If } a>0, \text{ then } \lim_{n\to\infty}\sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a}-1) = \frac{2\alpha}{e}\ln a.$$

Proof.

$$\text{Since } \sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a}-1) = \sqrt[n]{(2n-1)!!F_n}\left(e^{\frac{\ln a}{n}}-1\right) =$$

$$= \sqrt[n]{(2n-1)!!F_n}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\frac{\ln a}{n} = \sqrt[n]{\frac{(2n-1)!!F_n}{n^n}}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\ln a,$$

$$\text{we have } \lim_{n\to\infty}\sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a}-1) = \ln a\cdot 1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}} =$$

$$= \ln a\cdot\lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!F_n} = \ln a\cdot\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{2\alpha}{e}\ln a.$$

□

Theorem 7.

If $a>0$ and $(b_n)_{n\geq 1}$ is a positive real sequence with $\lim_{n\to\infty}\frac{b_{n+1}}{nb_n} = b > 0$, then

$$\lim_{n\to\infty}\sqrt[n]{b_nF_n}(\sqrt[n]{a}-1) = \frac{\alpha b}{e}\ln a.$$

Proof.

$$\text{Since } \sqrt[n]{b_nF_n}(\sqrt[n]{a}-1) = \sqrt[n]{b_nF_n}\left(e^{\frac{\ln a}{n}}-1\right) = \sqrt[n]{b_nF_n}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\frac{\ln a}{n} =$$

$$= \sqrt[n]{\frac{b_nF_n}{n^n}}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\ln a, \text{ we have } \lim_{n\to\infty}\sqrt[n]{n!F_n}(\sqrt[n]{a}-1) = \ln a\cdot 1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{b_nF_n}{n^n}} =$$

$$= \ln a\cdot\lim_{n\to\infty}\frac{b_{n+1}F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_nF_n} = \ln a\cdot\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\cdot\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{\alpha b}{e}\ln a.$$

□

Theorem 8.

$$\lim_{n\to\infty}n^2\sqrt[n]{n!F_n}\sin\frac{1}{n^3} = \frac{\alpha}{e}.$$

Proof.

$$\text{Since } n^2\sqrt[n]{n!F_n}\sin\frac{1}{n^3} = \frac{\sqrt[n]{n!F_n}}{n}\cdot n^3\cdot\sin\frac{1}{n^3} = \frac{\sin\frac{1}{n^3}}{\frac{1}{n^3}}\cdot\sqrt[n]{\frac{n!F_n}{n^n}}, \text{ then}$$

$$\lim_{n\to\infty} n^2\sqrt[n]{n!F_n}\sin\frac{1}{n^3} = 1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n!F_n}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!F_n} =$$

$$= \lim_{n\to\infty}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{\alpha}{e}.$$

□

Theorem 9.

$$\lim_{n\to\infty} n\sqrt[n]{(2n-1)!!F_n}\sin\frac{1}{n^2} = \frac{2\alpha}{e}.$$

Proof.

$$\text{Since } n\sqrt[n]{(2n-1)!!F_n}\sin\frac{1}{n^2} = \frac{\sqrt[n]{(2n-1)!!F_n}}{n}\cdot n^2\cdot\sin\frac{1}{n^2} = \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}\cdot\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}},$$

$$\text{then } \lim_{n\to\infty} n\sqrt[n]{n!F_n}\sin\frac{1}{n^2} = 1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}} =$$

$$= \lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!F_n} = \lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{2\alpha}{e}.$$

□

References

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