ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$, then prove that:

$$(a+b)\left(\frac{1}{\sqrt{a^2-ab+2b^2}}+\frac{1}{\sqrt{b^2-ab+2a^2}}\right)\le 2\sqrt{2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$(a+b)\left(\frac{1}{\sqrt{a^2-ab+2b^2}}+\frac{1}{\sqrt{b^2-ab+2a^2}}\right)$$

$$=(a+b)\left(\frac{1}{\sqrt{\sqrt{a^2-ab+b^2}^2+b^2}}+\frac{1}{\sqrt{\sqrt{a^2-ab+b^2}^2+a^2}}\right)$$

$$\le(a+b)\left(\frac{1}{\sqrt{\frac{1}{2}\left(\sqrt{a^2-ab+b^2}+b\right)^2}}+\frac{1}{\sqrt{\frac{1}{2}\left(\sqrt{a^2-ab+b^2}+a\right)^2}}\right)$$

$$=\sqrt{2}(a+b)\left(\frac{1}{\sqrt{a^2-ab+b^2}+b}+\frac{1}{\sqrt{a^2-ab+b^2}+a}\right)$$

$$=\frac{\sqrt{2}(a+b)\left(a+b+2\sqrt{a^2-ab+b^2}\right)}{a^2-ab+b^2+ab+(a+b).\sqrt{a^2-ab+b^2}}$$

$$=\frac{2\sqrt{2}\left((a+b)^2+2(a+b).\sqrt{a^2-ab+b^2}\right)}{2(a^2+b^2)+2(a+b).\sqrt{a^2-ab+b^2}}$$

$$\le\frac{2\sqrt{2}\left((a+b)^2+2(a+b).\sqrt{a^2-ab+b^2}\right)}{(a+b)^2+2(a+b).\sqrt{a^2-ab+b^2}}=2\sqrt{2},''='' \text{ iff } a=b \text{ (QED)}$$