ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } x, y, z > 0 \text{ such that}: \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le 3, \text{ then}:$$

$$\sqrt{\left(\frac{xy}{z}\right)^5}+\sqrt{\left(\frac{yz}{x}\right)^5}+\sqrt{\left(\frac{zx}{y}\right)^5} \ge \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

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$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le 3 \overset{\text{via A-G}}{\Rightarrow} 3 \ge 3.\sqrt[3]{\frac{1}{x}.\frac{1}{y}.\frac{1}{z}} \Rightarrow xyz \ge 1 \to (1)$$

$$\text{Now,}\left(\sqrt{\left(\frac{xy}{z}\right)^5}+\sqrt{\left(\frac{yz}{x}\right)^5}+\sqrt{\left(\frac{zx}{y}\right)^5}\right)^2 = \sum_{\text{cyc}}\left(\frac{yz}{x}\right)^5 + 2\sum_{\text{cyc}}\sqrt{\left(\frac{xy}{z}\right)^5}.\sqrt{\left(\frac{zx}{y}\right)^5}$$

$$\overset{\because yz \ge \frac{1}{x} \text{ and analogs ... via (1)}}{\ge} \sum_{\text{cyc}}\frac{1}{x^{10}} + 2\sum_{\text{cyc}} x^5 = \sum_{\text{cyc}}\left(\frac{1}{x^{10}}+x^5+x^5\right) \overset{\text{A-G}}{\ge}$$

$$\sum_{\text{cyc}} 3.\sqrt[3]{\frac{1}{x^{10}}.x^5.x^5} = 9 \ge \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2 \left(\because 3 \ge \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\therefore \sqrt{\left(\frac{xy}{z}\right)^5}+\sqrt{\left(\frac{yz}{x}\right)^5}+\sqrt{\left(\frac{zx}{y}\right)^5} \ge \frac{1}{x}+\frac{1}{y}+\frac{1}{z}, \text{''}=\text{'' iff } x=y=z=1 \text{ (QED)}$$