ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ *such that* : $a + b + c = 3$, then :

$$\frac{9}{2} + \sqrt{\left(\frac{ab}{c}\right)^5} + \sqrt{\left(\frac{bc}{a}\right)^5} + \sqrt{\left(\frac{ca}{b}\right)^5} \geq \frac{45}{2} \cdot \frac{abc}{ab + bc + ca}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} ab \overset{A-G}{\geq} 3.\sqrt[3]{a^2b^2c^2} \overset{?}{\geq} 3abc \Leftrightarrow abc \overset{?}{\leq} 1 \to \text{true} \because 3 = \sum_{cyc} a \overset{A-G}{\geq} 3.\sqrt[3]{abc}$$

$$\Rightarrow abc \leq 1 \Rightarrow \frac{abc}{ab + bc + ca} \leq \frac{1}{3} \Rightarrow \frac{45}{2} \cdot \frac{abc}{ab + bc + ca} - \frac{9}{2} \leq 3 \to (1)$$

$$\text{Let } \frac{bc}{a} = x, \frac{ca}{b} = y, \frac{ab}{c} = z \Rightarrow abc = xyz \Rightarrow a^2 = yz, b^2 = zx, c^2 = xy$$

$$\Rightarrow a = \sqrt{yz}, b = \sqrt{zx}, c = \sqrt{xy} \therefore \sum_{cyc} \sqrt{xy} = 3 \to (2)\ (\because a + b + c = 3)$$

$$\sqrt{\left(\frac{ab}{c}\right)^5} + \sqrt{\left(\frac{bc}{a}\right)^5} + \sqrt{\left(\frac{ca}{b}\right)^5} = \sum_{cyc} x^{\frac{5}{2}} \overset{?}{\geq} 3 \Leftrightarrow \sum_{cyc} u^5 \underset{(*)}{\overset{?}{\geq}} 3\ \left(u = \sqrt{x}, v = \sqrt{y}, w = \sqrt{z}\right)$$

$$\text{Now}, \sum_{cyc} u^5 \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} u^4\right) \cdot \sqrt{\left(\sum_{cyc} u\right)^2} \geq \frac{1}{9}\left(\sum_{cyc} u^2\right)^2 \cdot \sqrt{3\sum_{cyc} uv}$$

$$\geq \frac{1}{9}\left(\sum_{cyc} uv\right)^2 \cdot \sqrt{3\sum_{cyc} uv} = \frac{1}{9}\left(\sum_{cyc} \sqrt{xy}\right)^2 \cdot \sqrt{3\sum_{cyc} \sqrt{xy}} \overset{\text{via }(2)}{=} \frac{1}{9}.9.\sqrt{3.3} = 3$$

$$\Rightarrow (*) \text{ is true} \therefore \sqrt{\left(\frac{ab}{c}\right)^5} + \sqrt{\left(\frac{bc}{a}\right)^5} + \sqrt{\left(\frac{ca}{b}\right)^5} \geq 3 \overset{\text{via }(1)}{\geq} \frac{45}{2} \cdot \frac{abc}{ab + bc + ca} - \frac{9}{2}$$

$$\Rightarrow \frac{9}{2} + \sqrt{\left(\frac{ab}{c}\right)^5} + \sqrt{\left(\frac{bc}{a}\right)^5} + \sqrt{\left(\frac{ca}{b}\right)^5} \geq \frac{45}{2} \cdot \frac{abc}{ab + bc + ca},$$

$$'' ='' \text{ iff } a = b = c = 1\ (\text{QED})$$