ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then :

$$\sum_{cyc}(a^2 - 4\ln a) \geq \frac{3}{2}\left(\frac{a+b+c}{3}\right)^2 - 8.\sqrt{\frac{a+b+c}{3}} + \frac{65}{8}$$

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Let $f(x) = x^2 - 4\ln x\ \forall\, x > 0 \therefore f''(x) = \frac{4}{x^2} + 2 \Rightarrow f(x)$ is convex

$$\therefore \sum_{cyc}(a^2 - 4\ln a) \overset{\text{Jensen}}{\geq} 3\left(\left(\frac{\sum_{cyc} a}{3}\right)^2 - 4\ln\left(\frac{\sum_{cyc} a}{3}\right)\right) \overset{?}{>}$$

$$\frac{3}{2}\left(\frac{a+b+c}{3}\right)^2 - 8.\sqrt{\frac{a+b+c}{3}} + \frac{65}{8}$$

$$\Leftrightarrow \frac{3}{2}\left(\frac{\sum_{cyc} a}{3}\right)^2 + 8.\sqrt{\frac{\sum_{cyc} a}{3}} - \frac{65}{8} \overset{?}{>} 12\ln\left(\frac{\sum_{cyc} a}{3}\right)$$

$$\Leftrightarrow \frac{3}{2}.t^4 + 8t - \frac{65}{8} \overset{?}{>} 12t^2\left(t = \sqrt{\frac{\sum_{cyc} a}{3}}\right) \Leftrightarrow 12t^4 + 64t - 65 \boxed{\underset{(*)}{\overset{?}{>}}} 192\ln t$$

$$\text{Lem } F(m) = \ln m - \left(m - 1 - \frac{(m-1)^2}{2} + \frac{(m-1)^3}{3}\right) \forall\, m > 0$$

$\therefore F'(m) = \frac{(1-m)^3}{m}$. For $m \geq 1, F'(m) \leq 0 \Rightarrow F(m)$ is $\downarrow$ on $[1, \infty) \Rightarrow F(m) \leq F(1) = 0$ *and* for $0 < m \leq 1, F'(m) \geq 0 \Rightarrow F(m)$ is $\uparrow$ on $(0, 1] \Rightarrow F(m) \leq F(1) = 0$

$$\therefore F(m) \leq 0\ \forall\, m > 0 \Rightarrow \ln m \leq m - 1 - \frac{(m-1)^2}{2} + \frac{(m-1)^3}{3}\ \forall\, m > 0 \to (1)$$

$\boxed{\textit{Case 1}}$ $0 < t < \frac{3}{5}$ *and then* : $192\ln t < 192\ln\frac{3}{5} \approx -98.0785 < -98$ $\overset{?}{<} 12t^4 + 64t - 65 \Leftrightarrow 12t^4 + 64t + 33 \overset{?}{>} 0 \to$ true $\Rightarrow (*)$ is true

$\boxed{\textit{Case 2}}$ $t \geq \frac{3}{5}$ *and* $192\ln t \leq 192(t-1) - 96(t-1)^2 + 64(t-1)^3 \overset{?}{<}$ $12t^4 + 64t - 65 \Leftrightarrow 12t^4 - 64t^3 + 288t^2 - 512t + 287 \overset{?}{>} 0$

$$\Leftrightarrow 48t^4 - 256t^3 + 1152t^2 - 2048t + 1148 \overset{?}{>} 0$$

$$\Leftrightarrow \frac{1}{16}(48t^2 - 136t + 737)(4t-5)^2 + 7\left(t - \frac{57}{112}\right) \underset{(\bullet)}{\overset{?}{>}} 0$$

Now, discriminant of : $48t^2 - 136t + 737 = 136^2 - 192 * 737 = -123008 < 0$

$\Rightarrow 48t^2 - 136t + 737 > 0$ *and* $\because t \geq \frac{3}{5} \therefore t - \frac{57}{112} > 0 \therefore$ LHS of $(\bullet) > 0$

$\Rightarrow (\bullet) \Rightarrow (*)$ is true $\therefore$ combining both *cases*, $(*)$ is true $\forall\, t > 0$

$$\therefore \sum_{cyc}(a^2 - 4\ln a) > \frac{3}{2}\left(\frac{a+b+c}{3}\right)^2 - 8.\sqrt{\frac{a+b+c}{3}} + \frac{65}{8}\ \text{(QED)}$$