ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0, x+y+z=1$ then:

$$\frac{1}{x}\left(\frac{y^3}{x+z}+\frac{z^3}{x+y}\right)+\frac{1}{y}\left(\frac{x^3}{y+z}+\frac{z^3}{y+x}\right)+\frac{1}{z}\left(\frac{x^3}{z+y}+\frac{y^3}{z+x}\right)\geq 1$$

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Let's group the expression on the left in another form:

$$\underbrace{\frac{1}{x}\cdot\frac{y^3}{x+z}}_{II}+\underbrace{\frac{1}{x}\cdot\frac{z^3}{x+y}}_{III}+\underbrace{\frac{1}{y}\cdot\frac{x^3}{y+z}}_{I}+\underbrace{\frac{1}{y}\cdot\frac{z^3}{x+y}}_{III}+\underbrace{\frac{1}{z}\cdot\frac{x^3}{y+z}}_{I}+\underbrace{\frac{1}{z}\cdot\frac{y^3}{x+z}}_{II}\geq 1$$

$$\underbrace{\left(\frac{1}{y}\cdot\frac{x^3}{y+z}+\frac{1}{z}\cdot\frac{x^3}{y+z}\right)}_{I}+\underbrace{\left(\frac{1}{x}\cdot\frac{y^3}{x+z}+\frac{1}{z}\cdot\frac{y^3}{x+z}\right)}_{II}+\underbrace{\left(\frac{1}{x}\cdot\frac{z^3}{x+y}+\frac{1}{y}\cdot\frac{x^3}{y+z}\right)}_{III}\geq 1$$

$$\frac{x^3}{y+z}\cdot\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{y^3}{x+z}\cdot\left(\frac{1}{x}+\frac{1}{z}\right)+\frac{z^3}{x+y}\cdot\left(\frac{1}{x}+\frac{1}{y}\right)\geq 1$$

$$\frac{x^3}{y+z}\cdot\frac{y+z}{yz}+\frac{y^3}{x+z}\cdot\frac{x+z}{xz}+\frac{z^3}{x+y}\cdot\frac{x+y}{xy}\geq 1$$

$$\frac{x^3}{yz}+\frac{y^3}{xz}+\frac{z^3}{xy}\geq 1\quad (Both\ side\ mutliply\ xyz>0)$$

From here

$$x^4+y^4+z^4\geq xyz\ (*)$$

Because:

$$x^4+y^4+z^4=\left(\frac{1}{2}x^4+\frac{1}{2}y^4\right)+\left(\frac{1}{2}x^4+\frac{1}{2}z^4\right)+\left(\frac{1}{2}y^4+\frac{1}{2}z^4\right)\quad (1)$$

$$\left.\begin{aligned}\frac{1}{2}x^4+\frac{1}{2}y^4&\overset{A-G}{\geq}2\sqrt{\frac{1}{2}x^4\frac{1}{2}y^4}=x^2y^2\\ \frac{1}{2}x^4+\frac{1}{2}z^4&\overset{A-G}{\geq}2\sqrt{\frac{1}{2}x^4\frac{1}{2}z^4}=x^2z^2\\ \frac{1}{2}y^4+\frac{1}{2}z^4&\overset{A-G}{\geq}2\sqrt{\frac{1}{2}y^4\frac{1}{2}z^4}=y^2z^2\end{aligned}\right\}(2)$$

(2) using in (1)

$$x^4+y^4+z^4\geq x^2y^2+x^2z^2+y^2z^2=\left(\frac{1}{2}x^2y^2+\frac{1}{2}y^2z^2\right)+\left(\frac{1}{2}y^2z^2+\frac{1}{2}x^2z^2\right)+\left(\frac{1}{2}x^2y^2+\frac{1}{2}x^2z^2\right)\quad (3)$$

$$\left.\begin{aligned}
\frac{1}{2}x^2y^2+\frac{1}{2}y^2z^2 &\overset{A-G}{\geq} 2\sqrt{\frac{1}{2}x^2y^2\frac{1}{2}y^2z^2}=xy^2z\\
\frac{1}{2}y^2z^2+\frac{1}{2}x^2z^2 &\overset{A-G}{\geq} 2\sqrt{\frac{1}{2}y^2z^2\frac{1}{2}x^2z^2}=xyz^2\\
\frac{1}{2}x^2y^2+\frac{1}{2}x^2z^2 &\overset{A-G}{\geq} 2\sqrt{\frac{1}{2}x^2y^2\frac{1}{2}x^2z^2}=x^2yz
\end{aligned}\right\}(4)$$

(4) using in (3)

$$x^4+y^4+z^4\geq xy^2z+xyz^2+x^2yz=xyz(x+y+z)=xyz$$

So , $x^4+y^4+z^4\geq xyz\ (*)$

Equality holds if $x=y=z=\frac{1}{3}$