ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $xyz = 1$, then prove that :

$$\frac{(\sqrt{2}x)^2}{(1+xz)(1+xy)}+\frac{(\sqrt{2}y)^2}{(1+yz)(1+xy)}+\frac{(\sqrt{2}z)^2}{(1+xz)(1+yz)}\ge\frac{3}{2}$$

Proposed by Lamiye Quliyeva-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x^2}{(1+xz)(1+xy)}\overset{xyz=1}{=}\frac{x^2}{(xyz+xz)(xyz+xy)}=\frac{1}{yz(y+1)(z+1)}$$
$$\overset{xyz=1}{=}\frac{x(x+1)}{(x+1)(y+1)(z+1)} \text{ and analogs}$$
$$\therefore \frac{x^2}{(1+xz)(1+xy)}+\frac{y^2}{(1+yz)(1+xy)}+\frac{z^2}{(1+xz)(1+yz)}$$
$$=\sum_{cyc}\frac{x(x+1)}{(x+1)(y+1)(z+1)}\overset{?}{\ge}\frac{3}{4}\Leftrightarrow 4\sum_{cyc}x^2+\sum_{cyc}x\underset{(*)}{\overset{?}{\ge}}3xyz+3+3\sum_{cyc}xy$$
$$\text{Now, } 4\sum_{cyc}x^2+\sum_{cyc}x\ge\sum_{cyc}x^2+3\sum_{cyc}xy+\sum_{cyc}x\overset{A-G}{\ge}$$
$$3\sqrt[3]{x^2y^2z^2}+3\sum_{cyc}xy+3\sqrt[3]{xyz}\overset{xyz=1}{=}3+3+3\sum_{cyc}xy\overset{xyz=1}{=}3xyz+3+3\sum_{cyc}xy$$
$$\Rightarrow (*) \text{ is true } \therefore \frac{x^2}{(1+xz)(1+xy)}+\frac{y^2}{(1+yz)(1+xy)}+\frac{z^2}{(1+xz)(1+yz)}\ge\frac{3}{4}$$
$$\Rightarrow\frac{(\sqrt{2}x)^2}{(1+xz)(1+xy)}+\frac{(\sqrt{2}y)^2}{(1+yz)(1+xy)}+\frac{(\sqrt{2}z)^2}{(1+xz)(1+yz)}\ge\frac{3}{2}$$
$$\forall\, x,y,z>0 \,|\, xyz=1, ''='' \text{ iff } x=y=z=1$$

Solution 2 by Pham Duc Nam-Vietnam

$$\frac{2x^2}{(1+xz)(1+xy)}+\frac{2y^2}{(1+yz)(1+xy)}+\frac{2z^2}{(1+xz)(1+yz)}\ge\frac{3}{2}$$

$$*x+y+z\ge 3\sqrt[3]{xyz}=3. \text{ And}:$$
$$(x+y+z)^2\ge 3(xy+yz+xz)\Longrightarrow 2(xy+yz+xz)\le\frac{2(x+y+z)^2}{3}$$

$$*(1+xz)(1+xy)=xxyz+xy+xz+1=x+xy+xz+xyz=x(1+y)(1+z)\Rightarrow$$

$$(1+yz)(1+xy)=y(1+x)(1+z),(1+xz)(1+yz)=z(1+x)(1+y)\Rightarrow$$

$$\frac{2x^2}{(1+xz)(1+xy)}+\frac{2y^2}{(1+xy)(1+yz)}+\frac{2z^2}{(1+xz)(1+yz)}\overset{Bergstrom's}{\geq}$$

$$\geq\frac{2(x+y+z)^2}{x(1+y)(1+z)+y(1+x)(1+z)+z(1+x)(1+y)}=$$

$$=\frac{2(x+y+z)^2}{x+y+z+3xyz+2(xy+yz+xz)}\geq\frac{2(x+y+z)^2}{\frac{2}{3}(x+y+z)^2+(x+y+z)+3}$$

$$\because f(t)=\frac{2t^2}{\frac{2}{3}t^2+t+3},t\geq 3,f''=\frac{18t(t+6)}{(2t^2+3t+9)^2}>0\forall t\geq 3\Rightarrow$$

$$\frac{2x^2}{(1+xz)(1+xy)}+\frac{2y^2}{(1+xy)(1+yz)}+\frac{2z^2}{(1+xz)(1+yz)}\geq\frac{2}{3}$$

Equality holds iff : x=y=z=1