ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} > 0$ and $a + \mathrm{b} = 2$, then :

$$\frac{1}{a^a} + \frac{1}{\mathrm{b}^\mathrm{b}} \le 2$$

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$$\text{Let } f(x) = (2-x).x^x - 1\ \forall\, x \in (0,1] \text{ and then :}$$

$$f\,'(x) = -x^x\big((x-2)\ln x + x - 1\big) = x^x.(2-x)\ln x - x^x.(x-1)$$

$$\le x^x.(2-x)(x-1) - x^x.(x-1) = x^x.(x-1)(2-x-1) = -x^x.(x-1)^2 \le 0$$

$$\Rightarrow f\,'(x) \le 0\ \forall\, x \in (0,1] \Rightarrow f(x) \text{ is} \downarrow \text{on } (0,1] \Rightarrow f(x) \ge f(1) = 0\ \forall\, x \in (0,1]$$

$$\therefore (2-x).x^x - 1 \ge 0\ \forall\, x \in (0,1] \to (1)$$

$$\boxed{\text{Case 1}}\ 0 < a \le 1 \text{ and we have : } \frac{1}{\mathrm{b}^\mathrm{b}} = \frac{\mathrm{b}}{\mathrm{b}.\mathrm{b}^\mathrm{b}} = \frac{\mathrm{b}^{1-\mathrm{b}}}{\mathrm{b}} = \frac{\big(1+(\mathrm{b}-1)\big)^{1-\mathrm{b}}}{\mathrm{b}}$$

$$\overset{\text{Bernoulli}}{\le} \frac{1+(\mathrm{b}-1)(1-\mathrm{b})}{\mathrm{b}} = 2-\mathrm{b} \Rightarrow \frac{1}{a^a} + \frac{1}{\mathrm{b}^\mathrm{b}} \le \frac{1}{a^a} + 2 - \mathrm{b} \overset{?}{\le} 2 \Leftrightarrow \frac{1}{a^a} \overset{?}{\le} \mathrm{b}$$

$$= 2 - a \Leftrightarrow (2-a).a^a - 1 \overset{?}{\ge} 0 \to \text{true via } (1) \therefore \frac{1}{a^a} + \frac{1}{\mathrm{b}^\mathrm{b}} \le 2$$

$$\boxed{\text{Case 2}}\ 1 \le a < 2 \text{ and then : } 0 < \mathrm{b} \le 1 \text{ and we have : } \frac{1}{a^a} = \frac{a}{a.a^a} = \frac{a^{1-a}}{a}$$

$$= \frac{\big(1+(a-1)\big)^{1-a}}{a} \overset{\text{Bernoulli}}{\le} \frac{1+(a-1)(1-a)}{a} = 2 - a$$

$$\Rightarrow \frac{1}{a^a} + \frac{1}{\mathrm{b}^\mathrm{b}} \le 2 - a + \frac{1}{\mathrm{b}^\mathrm{b}} \overset{?}{\le} 2 \Leftrightarrow \frac{1}{\mathrm{b}^\mathrm{b}} \overset{?}{\le} a = 2 - \mathrm{b} \Leftrightarrow (2-\mathrm{b}).\mathrm{b}^\mathrm{b} - 1 \overset{?}{\ge} 0$$

$$\to \text{true via } (1) \therefore \frac{1}{a^a} + \frac{1}{\mathrm{b}^\mathrm{b}} \le 2 \therefore \text{combining cases } (1) \text{ and } (2), \frac{1}{a^a} + \frac{1}{\mathrm{b}^\mathrm{b}} \le 2$$

$$\forall\, a, \mathrm{b} > 0 \,\big|\, a + \mathrm{b} = 2, \text{''} = \text{''} \text{ iff } a = \mathrm{b} = 1 \text{ (QED)}$$